Numerical Analysis Module 2
Fundamentals of Vector Spaces

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1 Introduction

When we begin to use the concept of *vectors* for formulating mathematical models for physical systems, we start with the concept of a vector in the three dimensional coordinate space. From the mathematical viewpoint, the three dimensional space can be looked upon as a set of *objects*, called *vectors*, which satisfy certain generic properties. While working with mathematical modeling we need to deal with variety of such sets containing different types of *objects*. It is possible to *distill* essential properties satisfied by all the vectors in the three dimensional vector space and develop a more general concept of a *vector space*, which is a set of objects that satisfy these generic properties. Such a generalization can provide a unified view of problem formulations and the solution techniques. Generalization of the concept of the vector and the three dimensional vector space to any general set is not sufficient. To work with these sets of generalized vectors, we also need to generalize various algebraic and geometric concepts, such as *magnitude* of a vector, convergence of a sequence of vectors, limit, angle between two vectors, orthogonality etc., on these sets. Understanding the fundamentals of vector spaces helps in developing a unified view of many seemingly different numerical schemes. In this module, fundamentals of vector spaces are briefly introduced. A more detailed treatment of these topics can be found in Luenberger [2] and Kreyzig [1].

A word of advice before we begin to study these grand generalizations. While dealing with the generalization of geometric notions in three dimensions to more general vector spaces, it is difficult to *visualize* vectors and surfaces as we can do in the three dimensional vector space. However, if you understand the geometrical concepts in the three dimensional space well, then you can develop an understanding of the corresponding concept in any general vector space. In short, it is enough to know your school geometry well. We are only building qualitatively similar structures on the other sets.

2 Vector Spaces

The concept of a vector space will now be formally introduced. This requires the concept of *closure* and *field.*

Definition 1 ***(Closure)*** *A set is said to be closed under an operation when any two elements of the set subject to the operation yields a third element belonging to the same set.*

Example 2 *The set of integers is closed under addition, multiplication and subtraction. However, this set is not closed under division.*

Example 3 *The set of real numbers (R) and the set of complex numbers (C) are closed under addition, subtraction, multiplication and division.*

Definition 4 ***(Field)*** *A field is a set of elements closed under addition, subtraction, multiplication and division.*

Example 5 *The set of real numbers (R) and the set of complex numbers (C) are scalar fields. However, the set of integers is not a field.*

A vector space is a set of elements, which is closed under addition and scalar multiplication. Thus, associated with every vector space is a set of scalars F (also called as *scalar field* or *coefficient field*) used to define scalar multiplication on the space. In functional analysis, the scalars will be always taken to be the set of real numbers (R) or complex numbers (C).

Definition 6 ***(Vector Space):*** *A vector space X is a set of elements called vectors and scalar field F together with two operations. The first operation is called addition which associates with any two vectors $\mathbf{x}, \mathbf{y} \in X$ a vector $\mathbf{x} + \mathbf{y} \in X$, the sum of $\mathbf{x}$ and $\mathbf{y}$. The second operation is called scalar multiplication, which associates with any vector $\mathbf{x} \in X$ and any scalar α a vector $\alpha\mathbf{x}$ (a scalar multiple of $\mathbf{x}$ by α).*

Thus, when X is a linear vector space, given any vectors $\mathbf{x}, \mathbf{y} \in X$ and any scalars $\alpha, \beta \in R$, the element $\alpha\mathbf{x} + \beta\mathbf{y} \in X$. This implies that the well known parallelogram law in three dimensions also holds true in any vector space. Thus, given a vector space X and scalar field F, the fallowing properties hold for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ and any scalars $\alpha, \beta \in F$:

1. Commutative law: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
2. Associative law: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
3. There exists a null vector $\overline{0}$ such that $\mathbf{x} + \overline{0} = \mathbf{x}$ for all $\mathbf{x} \in X$
4. Distributive laws: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$, $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ and $\alpha\beta(\mathbf{x}) = \alpha(\beta\mathbf{x})$
5. $\alpha\mathbf{x} = \overline{\mathbf{0}}$ when $\alpha = 0$ and $\alpha\mathbf{x} = \mathbf{x}$ when $\alpha = 1$.
6. For convenience $-1\mathbf{x}$ is defined as $-\mathbf{x}$ and called as negative of a vector. We have $\mathbf{x} + (-\mathbf{x}) = \overline{\mathbf{0}}$, where $\overline{\mathbf{0}}$ represents zero vector in X.

Example 7 $(X \equiv R^n, F \equiv R)$: *$n-$ dimensional real coordinate space. A typical element $\mathbf{x} \in X$ can be expressed as*

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & ..... & x_n \end{bmatrix}^T$$

where x_i denotes the i'th element of the vector.

Example 8 *$(X \equiv C^n, F \equiv C)$: $n-$ dimensional complex coordinate space.*

Example 9 *$(X \equiv R^n, F \equiv C)$: This combination of set X and scalar field F does not form a vector space. For any $\mathbf{x} \in X$ and any $\alpha \in C$ the vector $\alpha\mathbf{x} \notin X$.*

Example 10 *$(X \equiv l_\infty, F \equiv R)$: Set of all infinite sequence of real numbers. A typical vector $\mathbf{x}$ of this space has form $\mathbf{x} = (\zeta_1, \zeta_2, ..., \zeta_k, ...)$.*

Example 11 *$(X \equiv C[a,b],\ F \equiv R)$: Set of all continuous functions over an interval $[a,b]$ forms a vector space. We write $\mathbf{x} = \mathbf{y}$ if $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \in [a,b]$ The null vector $\overline{\mathbf{0}}$ in this space is a function which is zero every where on $[a,b]$,i.e.*

$$f(t) = 0 \text{ for all } t \in [a,b]$$

If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are vectors from this space and α is real scalar, then $(\mathbf{x}+\mathbf{y})(t) = \mathbf{x}(t) + \mathbf{y}(t)$ and $(\alpha\mathbf{x})(t) = \alpha\mathbf{x}(t)$ are also elements of $C[a,b]$.

Example 12 *$\left(X \equiv C^{(n)}[a,b],\ F \equiv R\right)$: Set of all continuous and n times differentiable functions over an interval $[a,b]$ forms a vector space.*

Example 13 *$X \equiv$ set of all real valued polynomial functions defined on interval $[a,b]$ together with $F \equiv R$ forms a vector space.*

Example 14 *The set of all functions $\{f(t) : t \in [a,b]\}$ for which*

$$\int_a^b |f(t)|^p \, dt < \infty$$

holds is a linear space L_p.

Example 15 *$(X \equiv R^m \times R^n,\ F \equiv R)$: Here we consider the set of all $m \times n$ matrices with real elements. It is easy to see that, if $A, B \in X$, then $\alpha A + \beta B \in X$ and X is a vector space. Note that a* vector *in this space is a $m \times n$ matrix and the null vector corresponds to $m \times n$ null matrix.*

In three dimension, we often have to work with a line or a plane passing through the origin, which form a subspace of the three dimensional space. The concept of a sub-space can be generalized as follows.

Definition 16 ***(Subspace):*** *A non-empty subset M of a vector space X is called subspace of X if every vector $\alpha\mathbf{x}+\beta\mathbf{y}$ is in M wherever $\mathbf{x}$ and $\mathbf{y}$ are both in M. Every subspace always contains the null vector, I.e. the origin of the space $\mathbf{x}$.*

Thus, the fundamental property of objects (elements) in a vector space is that they can be constructed by simply adding other elements in the space. This property is formally defined as follows.

Definition 17 ***(Linear Combination):*** *A linear combination of vectors* $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ...\mathbf{x}^{(m)}$ *in a vector space is of the form* $\alpha_1\mathbf{x}^{(1)} + \alpha_2\mathbf{x}^{(2)} + ... + \alpha_m\mathbf{x}^{(m)}$ *where* $(\alpha_1, ...\alpha_m)$ *are scalars.*

Note that we are dealing with set of vectors

$$\left\{\mathbf{x}^{(k)} : k = 1, 2, ...m.\right\} \tag{1}$$

The individual elements in the set are indexed using superscript (k). Now, if $X = R^n$ and $\mathbf{x}^{(k)} \in R^n$ represents k'th vector in the set, then it is a vector with n components which are represented as follows

$$\mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} & x_2^{(k)} & .... & x_n^{(k)} \end{bmatrix}^T \tag{2}$$

Similarly, if $X = l_\infty$ and $\mathbf{x}^{(k)} \in l_\infty$ represents k'th vector in the set, then $\mathbf{x}^{(k)}$ represents an infinite sequence with elements denoted as follows

$$\mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} & .... & x_i^{(k)} & ...... \end{bmatrix}^T \tag{3}$$

Definition 18 ***(Span of Set of Vectors):*** *Let S be a subset of vector space X. The set generated by all possible linear combinations of elements of S is called as span of S and denoted as $[S]$. Span of S is a subspace of X.*

Definition 19 ***(Linear Dependence):*** *A vector* $\mathbf{x}$ *is said to be linearly dependent upon a set S of vectors if* $\mathbf{x}$ *can be expressed as a linear combination of vectors from S. Alternatively,* $\mathbf{x}$ *is linearly dependent upon S if* $\mathbf{x}$ *belongs to the span of S, i.e.* $\mathbf{x} \in [S]$. *A vector is said to be linearly independent of set S, if it not linearly dependent on S . A necessary and sufficient condition for the set of vectors* $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ...\mathbf{x}^{(m)}$ *to be linearly independent is that expression*

$$\sum_{i=1}^{m} \alpha_i \mathbf{x}^{(i)} = \overline{0} \tag{4}$$

implies that $\alpha_i = 0$ for all $i = 1, 2......m$.

Definition 20 ***(Basis):*** *A finite set S of linearly independent vectors is said to be basis for space X if S generates X i.e. $X = [S]$.*

Example 21 ***Basis, Span and Sub-spaces***

1. *Two dimensional plane passing through origin of R^3. For example, consider the set S of collection of all vectors*

$$\mathbf{x} = \alpha \mathbf{x}^{(1)} + \beta \mathbf{x}^{(2)}$$

where $\alpha, \beta \in R$ are arbitrary scalars and

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad and \quad \mathbf{x}^{(2)} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

i.e. $S = span\left\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right\}$. This set defines a plane passing through origin in R^3. Note that a plane which does not pass through the origin is not a sub-space. The origin must be included in the set for it to qualify as a sub-space.

2. *Let $S = \{\mathbf{v}\}$ where $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$ and let us define span of S as $[S] = \alpha\mathbf{v}$ where $\alpha \in R$ represents a scalar. Here, $[S]$ is one dimensional vector space and subspace of R^5*

3. *Let $S = \left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\right\}$ where*

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad ; \ \mathbf{v}^{(2)} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \tag{5}$$

Here span of S (i.e. $[S]$) is two dimensional subspace of R^5.

4. *Consider set of n^{th} order polynomials on interval $[0,1]$. A possible basis for this space is*

$$p^{(1)}(z) = 1; \ \ p^{(2)}(z) = z; \ p^{(3)}(z) = z^2, ..., p^{(n+1)}(z) = z^n \tag{6}$$

Any vector $p(t)$ from this space can be expressed as

$$\begin{aligned} p(z) &= \alpha_0 p^{(1)}(z) + \alpha_1 p^{(2)}(z) + ... + \alpha_n p^{(n+1)}(z) \\ &= \alpha_0 + \alpha_1 z + ... + \alpha_n z^n \end{aligned} \tag{7}$$

Note that $[S]$ in this case is $(n+1)$ dimensional subspace of $C[a,b]$.

5. *Consider set of continuous functions over interval, i.e. $C[-\pi, \pi]$. A well known basis for this space is*

$$x^{(0)}(z) = 1; \ x^{(1)}(z) = \cos(z); \ \ x^{(2)}(z) = \sin(z), \tag{8}$$

$$x^{(3)}(z) = \cos(2z), \ \ x^{(4)}(z) = \sin(2z), ... \tag{9}$$

It can be shown that $C[-\pi,\pi]$ is an infinite dimensional *vector space.*

6. *The set of all symmetric real valued $n \times n$ matrices is a subspace of the set of all real valued $n \times n$ matrices. This follows from the fact that matrix $\alpha\mathbf{A} + \beta\mathbf{B}$ is a real values symmetric matrix for arbitrary scalars $\alpha, \beta \in R$ when $\mathbf{A}^T = \mathbf{A}$ and $\mathbf{B}^T = \mathbf{B}$.*

Example 22 *Show that functions 1, exp(t), exp(2t), exp(3t) are linearly independent over any interval [a,b].*

Let us assume that vectors $(1, e^t, e^{2t}, e^{3t})$ are linearly dependent i.e. there are constants $(\alpha, \beta, \gamma, \delta)$, not all equal to zero, such that

$$\alpha + \beta e^t + \gamma e^{2t} + \delta e^{3t} = 0 \quad \textit{holds for all } t \in [a,b] \tag{10}$$

Taking derivative on both the sides, the above equality implies

$$e^t(\beta + 2\gamma e^t + 3\delta e^{2t}) = 0 \quad \textit{holds for all } t \in [a,b]$$

Since $e^t > 0$ holds for all $t \in [a,b]$, the above equation implies that

$$\beta + 2\gamma e^t + 3\delta e^{2t} = 0 \quad \textit{holds for all } t \in [a,b] \tag{11}$$

Taking derivative on both the sides, the above equality implies

$$e^t(2\gamma + 6\delta e^t) = 0 \quad \textit{holds for all } t \in [a,b]$$

which implies that

$$2\gamma + 6\delta e^t = 0 \quad \textit{holds for all } t \in [a,b] \tag{12}$$

$$\Rightarrow e^t = -\frac{\beta}{\alpha} \textit{holds for all } t \in [a,b]$$

which is absurd. Thus, equality (12) holds only for $\gamma = \delta = 0$ and vectors $(1, e^t)$ are linearly independent on any interval $[a,b]$. With $\gamma = \delta = 0$, equality (11) only when $\beta = 0$ and equality (10) holds only when $\alpha = 0$. Thus, vectors $(1, e^t, e^{2t}, e^{3t})$ are linearly independent.

Example 23 *Consider system of linear algebraic equations*

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{b}$$

Show that the set of all solutions of this equation for arbitrary vector $\mathbf{b}$ is same as R^3.

It is easy to see that matrix $\mathbf{A}$ has rank equal to three and columns (and rows) are linearly independent. Since the columns are linearly independent, a unique solution $\mathbf{x} \in R^3$ can be *found for any arbitrary vector* $\mathbf{b} \in R^3$*. Now, let us find a general solution* $\mathbf{x}$ *for an arbitrary vector* $\mathbf{b}$ *by computing* $\mathbf{A}^{-1}$ *as follows*

$$\begin{aligned}
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \mathbf{b} \\
&= b_1 \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + b_2 \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + b_3 \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\
&= b_1\mathbf{v}^{(1)} + b_2\mathbf{v}^{(2)} + b_3\mathbf{v}^{(3)}
\end{aligned}$$

By definition

$$b_1\mathbf{v}^{(1)} + b_2\mathbf{v}^{(2)} + b_3\mathbf{v}^{(3)} \in span\left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right\}$$

for an arbitrary $\mathbf{b} \in R^3$*, and, since vectors* $\left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right\}$ *are linearly independent, we have*

$$span\left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}\right\} = R^3$$

i.e. set of all possible solutions $\mathbf{x}$ *of the system of equations under considerations is identical to the entire space* R^3*.*

Example 24 *Consider the ODE-BVP*

$$\begin{aligned}
\frac{d^2u(z)}{dz^2} + \lambda^2 u(z) &= 0 \qquad for \quad 0 < z < 1 \\
B.C.1 \quad (at\ z &= 0) : u(0) = 0 \\
B.C.2 \quad (at\ z &= 1) : u(1) = 0
\end{aligned}$$

The general solution of this ODE-BVP, which satisfies the boundary conditions, is given by

$$u(z) = \alpha_1 \sin(\pi z) + \alpha_2 \sin(2\pi z) + \alpha_3 \sin(3\pi z) + ... = \sum_{i=1}^{\infty} \alpha_i \sin(i\pi z)$$

where $(\alpha_1, \alpha_2, ...) \in R$ *are arbitrary scalars. The set of vectors* $\{\sin(\pi z), \sin(2\pi z), \sin(3\pi z), ...\}$ *is linearly independent and form a basis for* $C^{(2)}[0,1]$*, i.e. the set of twice differentiable continuous functions in interval* $[0,1]$ *i.e.*

$$C^{(2)}[0,1] = span\left\{\sin(\pi z), \sin(2\pi z), \sin(3\pi z), ...\right\}$$

Example 25 *Consider system of linear algebraic equations*

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -2 & 4 \\ 2 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Show that solutions of this equation forms a two dimensional subspace of R^3.

It is easy to see that matrix $\mathbf{A}$ *has rank equal to one and columns (and rows) are linearly dependent. Thus, it is possible to obtain non-zero solutions to the above equation, which can be re-written as follows*

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Two possible solutions are

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad and \quad \mathbf{x}^{(2)} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

In fact, $\mathbf{x}^{(1)}$ *and* $\mathbf{x}^{(2)}$ *and linearly independent and any linear combination of these two vectors, i.e.*

$$\mathbf{x} = \alpha \mathbf{x}^{(1)} + \beta \mathbf{x}^{(2)}$$

for any scalars $(\alpha, \beta) \in R$ *satisfies*

$$\mathbf{A}\mathbf{x} = \mathbf{A}\left(\alpha \mathbf{x}^{(1)} + \beta \mathbf{x}^{(2)}\right) = \alpha \left(\mathbf{A}\mathbf{x}^{(1)}\right) + \beta \left(\mathbf{A}\mathbf{x}^{(2)}\right) = \overline{\mathbf{0}}.$$

Thus, the solutions can be represented by a set $S = span\left\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right\}$, *which forms a two dimensional subspace of R^3.*

Example 26 *Consider a third order linear ordinary differential equation*

$$\frac{d^3u}{dz^2} + 6\frac{d^2u}{dz^2} + 11\frac{du}{dz} + 6u = 0$$

defined over $C^{(3)}[0, 1]$, i.e. set of thrice differentiable continuous functions over $[0, 1]$. Show that the general solution of the ODE forms a 3 dimensional subspace of $C^{(3)}[0, 1]$.

Roots of the characteristic polynomial i.e.

$$p^3 + 6p^2 + 11p + 6 = 0$$

are $p = -1$, $p = -2$ and $p = -3$. Thus, general solution of the ODE can be written as

$$u(z) = \alpha e^{-z} + \beta e^{-2z} + \gamma e^{-3z}$$

where $(\alpha, \beta, \gamma) \in R$ are arbitrary scalars. Since vectors $\{e^{-z}, e^{-2z}, e^{-3z}\}$ are linearly independent, the set of solutions can be represented as $S = span\{e^{-z}, e^{-2z}, e^{-3z}\}$, which forms a three dimensional sub-space of $C^{(3)}[0, 1]$.

A vector space having finite basis (spanned by set of vectors with finite number of elements) is said to be finite dimensional. All other vector spaces are said to be infinite dimensional. We characterize a finite dimensional space by number of elements in a basis. Any two basis for a finite dimensional vector space contain the same number of elements.

Let X and Y be two vector spaces. Then their product space, denoted by $X \times Y$, is an ordered pair $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x} \in X,\ \mathbf{y} \in Y$. If $\mathbf{z}^{(1)} = (\mathbf{x}^{(1)}, \mathbf{y}^{(1)})$ and $\mathbf{z}^{(2)} = (\mathbf{x}^{(2)}, \mathbf{y}^{(2)})$ are two elements of $X \times Y$, then it is easy to show that $\alpha\mathbf{z}^{(1)} + \beta\mathbf{z}^{(2)} \in X \times Y$ for any scalars (α,β). Thus, product space is a linear vector space.

Example 27 *Let $X = C[a,b]$ and $Y = R$, then the product space $X \times Y = C[a,b] \times R$ forms a linear vector space. Such product spaces arise in the context of ordinary differential equations.*

3 Normed Linear Spaces and Banach Spaces

In three dimensional space, we use *lengths* or *magnitudes* to compare any two vectors. Generalization of the concept of length / magnitude of a vector in three dimensional vector space to an arbitrary vector space is achieved by defining a scalar valued function called *norm* of a vector.

Definition 28 ***(Normed Linear Vector Space):*** *A normed linear vector space is a vector space X on which there is defined a real valued function which maps each element $\mathbf{x} \in X$ into a real number $\|\mathbf{x}\|$ called norm of $\mathbf{x}$. The norm satisfies the fallowing axioms.*

1. $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in X$; $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \overline{0}$ (*zero vector*)
2. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for each $\mathbf{x}, \mathbf{y} \in X$. (*triangle inequality*).
3. $\|\alpha\mathbf{x}\| = |\alpha| \,.\, \|\mathbf{x}\|$ for all scalars α and each $\mathbf{x} \in X$

Example 29 ***Vector norms:***

1. *$(R^n, \|.\|_1)$:Euclidean space R^n with 1-norm: $\|\mathbf{x}\|_1 = \sum\limits_{i=1}^{N} |x_i|$*
2. *$(R^n, \|.\|_2)$:Euclidean space R^n with 2-norm:*

$$\|\mathbf{x}\|_2 = \left[\sum_{i=1}^{N} (x_i)^2\right]^{\frac{1}{2}}$$

3. $\left(R^n, \|.\|_p\right)$:*Euclidean space* R^n *with p-norm:*

$$\|\mathbf{x}\|_p = \left[\sum_{i=1}^{N} |x_i|^p\right]^{\frac{1}{p}} \tag{13}$$

where p *is a positive integer*

4. $(R^n, \|.\|_\infty)$:*Euclidean space* R^n *with* $\infty-$*norm:* $\|\mathbf{x}\|_\infty = \max |x_i|$

5. *n-dimensional complex space* (C^n) *with p-norm:*

$$\|\mathbf{x}\|_p = \left[\sum_{i=1}^{N} |x_i|^p\right]^{\frac{1}{p}} \tag{14}$$

, where p *is a positive integer*

6. *Space of infinite sequences* (l_∞) *with p-norm: An element in this space, say* $\mathbf{x} \in l_\infty$*, is an infinite sequence of numbers*

$$\mathbf{x} = \{x_{1,}x_2, ..., x_n, ...\} \tag{15}$$

such that p-norm defined as

$$\|\mathbf{x}\|_p = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} < \infty \tag{16}$$

is bounded for every $\mathbf{x} \in l_\infty$*, where* p *is an integer.*

7. $(C[a, b], \|\mathbf{x}(t)\|_\infty)$: *The normed linear space* $C[a, b]$ *together with infinite norm*

$$\|\mathbf{x}(t)\|_\infty = \begin{matrix} \max \\ a \le t \le b \end{matrix} \ |\mathbf{x}(t)| \tag{17}$$

It is easy to see that $\|\mathbf{x}(t)\|_\infty$ *defined above qualifies to be a norm*

$$max\, |\mathbf{x}(t) + \mathbf{y}(t)| \le \max[|\mathbf{x}(t)| + |\mathbf{y}(t)|] \quad \le \max |\mathbf{x}(t)| + \max |\mathbf{y}(t)| \tag{18}$$

$$max\, |\alpha \mathbf{x}(t)| = \max |\alpha|\, |\mathbf{x}(t)| = |\alpha| \max |\mathbf{x}(t)| \tag{19}$$

8. *Other types of norms, which can be defined on the set of continuous functions over* $[a, b]$ *are as follows*

$$\|\mathbf{x}(t)\|_1 = \int_a^b |\mathbf{x}(t)|\, dt \tag{20}$$

$$\|\mathbf{x}(t)\|_2 = \left[\int_a^b |\mathbf{x}(t)|^2 \, dt \right]^{\frac{1}{2}} \tag{21}$$

Example 30 *Determine whether(a)* $\max |df(t)/dt|$ *(b) max* $|\mathbf{x}(t)| + \max |\mathbf{x}'(t)|$ *(c)* $|\mathbf{x}(a)| + \max |\mathbf{x}'(t)|$ *and (d)* $|\mathbf{x}(a)| \max |\mathbf{x}(t)|$ *can serve as a valid definitions for norm in* $\mathbf{C}^{(2)}[a,b]$.

Solution: *(a)* $\max |df(t)/dt|$: *For this to be a norm function, Axiom 1 in the definition of the normed vector spaces requires*

$$\|f(t)\| = 0 \Rightarrow f(t) \text{ is the zero vector in } \mathbf{C}^{(2)}[a,b] \text{ i.e. } f(t) = 0 \quad \text{for all } t \in \; [a,b]$$

However, consider the constant function i.e. $g(t) = c$ *for all* $t \in \; [a,b]$ *where c is some non-zero value. It is easy to see that*

$$\max |dg(t)/dt| = 0$$

even when $g(t)$ *does not correspond to the zero vector. Thus, the above function violates Axiom 1 in the definition of a normed vector space and, consequently, cannot qualify as a norm.*

(b) max $|\mathbf{x}(t)| + \max |\mathbf{x}'(t)|$: *For any non-zero function* $\mathbf{x}(t) \in \mathbf{C}^{(2)}[a,b]$*, Axiom 1 is satisfied. Axiom 2 follows from the following inequality*

$$\begin{aligned}
\|\mathbf{x}(t) + \mathbf{y}(t)\| &= max\,|\mathbf{x}(t) + \mathbf{y}(t)| + \max |\mathbf{x}'(t) + \mathbf{y}'(t)| \\
&\leq [max\,|\mathbf{x}(t)| + max\,|\mathbf{y}(t)|] + [\max |\mathbf{x}'(t)| + \max |\mathbf{y}'(t)|] \\
&\leq [max\,|\mathbf{x}(t)| + \max |\mathbf{x}'(t)|] + [max\,|\mathbf{y}(t)| + \max |\mathbf{y}'(t)|] \\
&\leq \|\mathbf{x}(t)\| + \|\mathbf{y}(t)\|
\end{aligned}$$

It is easy to show that Axiom 3 is also satisfied for all scalars α. *Thus, given function defines a norm on* $C^{(2)}[a,b]$

(c) $|\mathbf{x}(a)| + \max |\mathbf{x}'(t)|$: *For any non-zero function* $\mathbf{x}(t) \in \mathbf{C}^{(2)}[a,b]$*, Axiom 1 is satisfied. Axiom 2 follows from the following inequality*

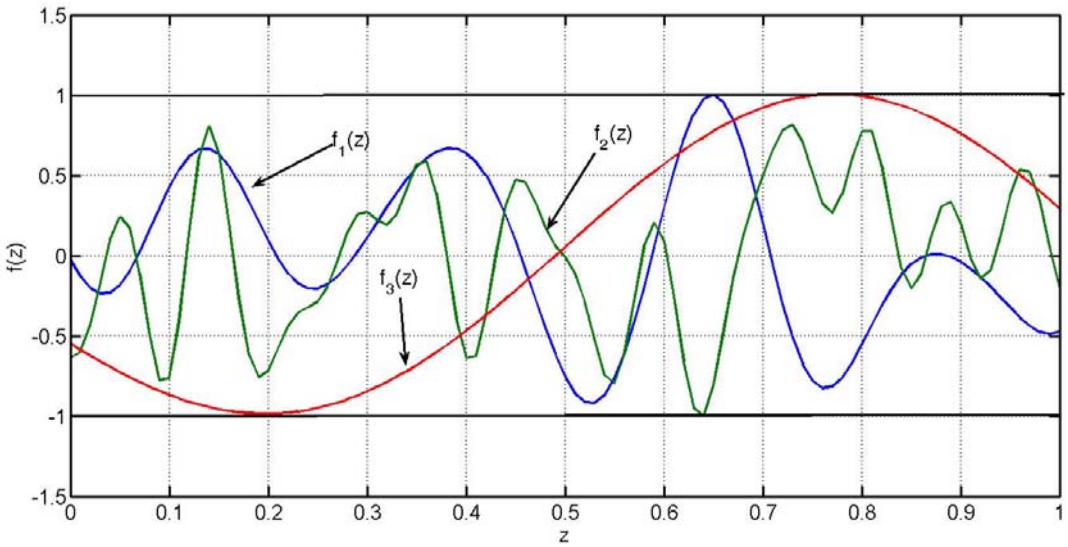
$$\begin{aligned}
\|\mathbf{x}(t) + \mathbf{y}(t)\| &= |\mathbf{x}(a) + \mathbf{y}(a)| + \max |\mathbf{x}'(t) + \mathbf{y}'(t)| \\
&\leq [|\mathbf{x}(a)| + |\mathbf{y}(a)|] + [\max |\mathbf{x}'(t)| + \max |\mathbf{y}'(t)|] \\
&\leq [|\mathbf{x}(a)| + \max |\mathbf{x}'(t)|] + [|\mathbf{y}(a)| + \max |\mathbf{y}'(t)|] \\
&\leq \|\mathbf{x}(t)\| + \|\mathbf{y}(t)\|
\end{aligned}$$

Axiom A3 is also satisfied for any α *as*

$$\begin{aligned}
\|\alpha \mathbf{x}(t)\| &= |\alpha \mathbf{x}(a)| + \max |\alpha \mathbf{x}'(t)| \\
&= |\alpha| \left[|\mathbf{x}(a)| + \max |\mathbf{x}'(t)| \right] \\
&= |\alpha| \,.\, \|\mathbf{x}\|
\end{aligned}$$

(d) $|\mathbf{x}(a)| \max |\mathbf{x}(t)|$: *Consider a non-zero function* $\mathbf{x}(t)$ *in* $C^{(2)}[a,b]$ *such that* $x(a) = 0$ *and* $\max |\mathbf{x}(t)| \neq 0$. *Then, Axiom 1 is not satisfied for all vector* $\mathbf{x}(t) \in C^{(2)}[a,b]$ *and the above function does not qualify to be a norm on* $C^{(2)}[a,b]$.

In a normed linear space X, the set of all vectors $\mathbf{x} \in X$ such that $\|\mathbf{x}-\overline{\mathbf{x}}\| \leq 1$ is called unit ball centered at $\overline{\mathbf{x}}$.A unit ball in $(R^2, \|.\|_2)$ is the set of all vectors in the circle with the origin at the center and radius equal to one while a unit ball in $(R^3, \|.\|_2)$ is the set of all points in the unit sphere with the origin at the center. Schematic representation of a unit ball in C[0,1] when maximum norm is used is shown in Figure 1. The unit ball in C[0,1] is set of all functions f(z) such that $|f(z)| \leq 1$ where $z \in [0,1]$.



Schematic representation of a unit ball in C[0,1]

Once we have defined a norm in a vector space, we can proceed to generalize the concept of convergence of a sequence of vector. Concept of convergence is central to all iterative numerical methods.

Definition 31 ***(Cauchy sequence):*** *A sequence* $\{\mathbf{x}^{(k)}\}$ *in normed linear space is said to be a Cauchy sequence if* $\|\mathbf{x}^{(n)} - \mathbf{x}^{(m)}\| \to 0$ *as* $n, m \to \infty$*.i.e. given an* $\varepsilon > 0$ *there exists an integer* N *such that* $\|\mathbf{x}^{(n)} - \mathbf{x}^{(m)}\| < \varepsilon$ *for all* $n, m \geq N$.

Definition 32 ***(Convergence):*** *In a normed linear space an infinite sequence of vectors* $\{\mathbf{x}^{(k)} : k = 1, 2, ...\}$ *is said to converge to a vector* $\mathbf{x}^*$ *if the sequence* $\{\|\mathbf{x}^* - \mathbf{x}^{(k)}\|, \ k = 1, 2, ...\}$ *of real numbers converges to zero. In this case we write* $\mathbf{x}^{(k)} \to \mathbf{x}^*$.

In particular, a sequence $\left\{\mathbf{x}^{(k)}\right\}$ in R^nconverges if and only if each component of the vector sequence converges. If a sequence converges, then its limit is unique.

Example 33 ***Convergent sequences:*** *Consider the sequence of vectors represented as*

$$\mathbf{x}^{(k)}=\begin{bmatrix} 1+(0.2)^k \\ -1+(0.9)^k \\ 3/\left(1+(-0.5)^k\right) \\ (0.8)^k \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix} \tag{22}$$

for k = 0, 1, 2,.... is a convergent sequence with respect to any p-norm defined on R^4. It can be shown that it is a Cauchy sequence. Note that each element of the vector converges to a limit in this case.

Every convergent sequence is a Cauchy sequence. Moreover, when we are working in R^n or C^n,all Cauchy sequences are convergent. However, all Cauchy sequences in a general vector space need not be convergent. Cauchy sequences in some vector spaces exhibit such strange behavior and this motivates the concept of completeness of a vector space.

Definition 34 ***(Banach Space):*** *A normed linear space X is said to be complete if every Cauchy sequence has a limit in X. A complete normed linear space is called Banach space.*

Examples of Banach spaces are

$$\left(\mathbf{R}^n, \|.\|_1\right),\ \left(\mathbf{R}^n, \|.\|_2\right), \left(\mathbf{R}^n, \|.\|_\infty\right)$$

$$\left(\mathbf{C}^n, \|.\|_1\right), \left(\mathbf{C}^n, \|.\|_2\right), \left(l_\infty, \|.\|_1\right), \left(l_\infty, \|.\|_2\right)$$

Concept of Banach spaces can be better understood if we consider an example of a vector space where a Cauchy sequence is not convergent, i.e. the space under consideration is an incomplete normed linear space. Note that, even if we find one Cauchy sequence in this space which does not converge, it is sufficient to prove that the space is not complete.

Example 35 *Let $X = (Q, \|.\|_1)$ i.e. set of rational numbers (Q) with scalar field also as the set of rational numbers (Q) and norm defined as*

$$\|x\|_1 = |x| \tag{23}$$

A vector in this space is a rational number. In this space, we can construct Cauchy sequences which do not converge to a rational numbers (or rather they converge to irrational numbers).

For example, the well known Cauchy sequence

$$\begin{aligned} x^{(1)} &= 1/1 \\ x^{(2)} &= 1/1 + 1/(2!) \\ &\ldots\ldots\ldots \\ x^{(n)} &= 1/1 + 1/(2!) + \ldots.. + 1/(n!) \end{aligned}$$

converges to e, which is an irrational number. Similarly, consider sequence

$$x^{(n+1)} = 4 - (1/x^{(n)})$$

Starting from initial point $x^{(0)} = 1$*, we can generate the sequence of rational numbers*

$$3/1, 11/3, 41/11, \ldots.$$

which converges to $2+\sqrt{3}$ *as* $n \rightarrow \infty$*.Thus, limits of the above sequences is outside the space* X *and the space is incomplete.*

Example 36 *Consider sequence of functions in the space of twice differentiable continuous functions* $C^{(2)}(-\infty, \infty)$

$$f^{(k)}(t) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(kt)$$

defined in interval $-\infty < t < \infty$*, for all integers* k*. The range of the function is (0,1). As* $k \rightarrow \infty$*, the sequence of continuous function converges to a discontinuous function*

$$\begin{aligned} u^{(*)}(t) &= 0 \qquad -\infty < t < 0 \\ &= 1 \qquad 0 < t < \infty \end{aligned}$$

Example 37 *Let* $X = (C[0,1], \|.\|_1)$ *i.e. space of continuous function on* $[0,1]$ *with one norm defined on it i.e.*

$$\|\mathbf{x}(t)\|_1 = \int_0^1 |\mathbf{x}(t)|\, dt \tag{24}$$

and let us define a sequence [2]

$$\mathbf{x}^{(n)}(t) = \left\{ \begin{array}{ll} 0 & (0 \leq t \leq (\frac{1}{2} - \frac{1}{n}) \\ n(t-\frac{1}{2})+1 & (\frac{1}{2} - \frac{1}{n}) \leq t \leq \frac{1}{2}) \\ 1 & (t \geq \frac{1}{2}) \end{array} \right\} \tag{25}$$

Each member is a continuous function and the sequence is Cauchy as

$$\left\|\mathbf{x}^{(n)} - \mathbf{x}^{(m)}\right\| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \rightarrow 0 \tag{26}$$

However, as can be observed from Figure 2, the sequence does not converge to a continuous function.

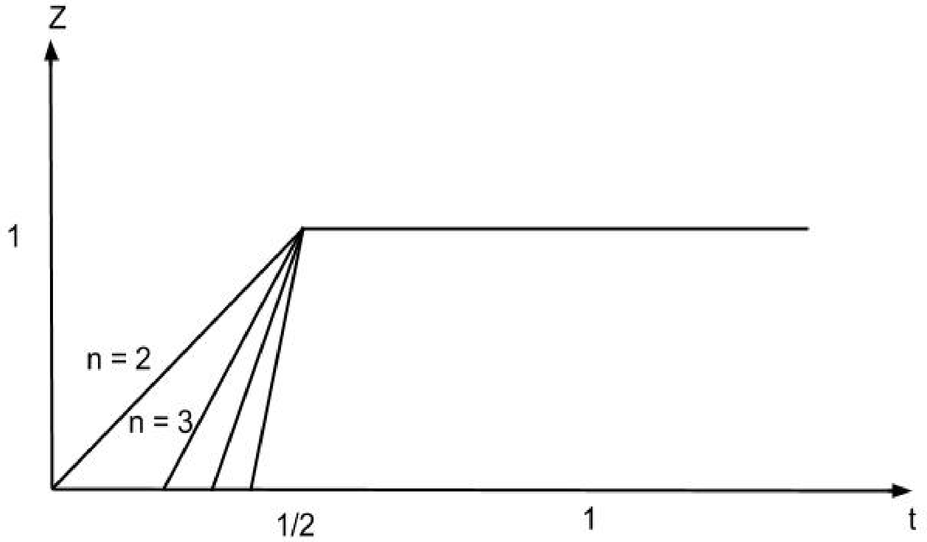


Figure 1: Sequence of continuous functions

The concepts of convergence, Cauchy sequences and completeness of space assume importance in the analysis of iterative numerical techniques. Any iterative numerical method generates a sequence of vectors and we have to assess whether the sequence is Cauchy to terminate the iterations. To a beginner, it may appear that the concept of incomplete vector space does not have much use in practice. It may be noted that, when we compute numerical solutions using any computer, we are working in finite dimensional incomplete vector spaces. In any computer with finite precision, any irrational number such as π or e, is approximated by an rational number due to finite precision. In fact, even if we want to find a solution in R^n, while using a finite precision computer to compute a solution, we actually end up working in Q^n and not in R^n.

4 Inner Product Spaces and Hilbert Spaces

Similar to magnitude / length of a vector, another important concept in three dimensional space that needs to be generalized is angle between any two vectors. Given any two unit vectors in R^3, say $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$,the angle between these two vectors is defined using inner (or dot) product of two vectors as

$$\cos(\theta) = (\widehat{\mathbf{x}})^T \widehat{\mathbf{y}} = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)^T \frac{\mathbf{y}}{\|\mathbf{y}\|_2} \tag{27}$$

$$= \widehat{x}_1\widehat{y}_1 + \widehat{x}_2\widehat{y}_2 + \widehat{x}_3\widehat{y}_3 \tag{28}$$

The fact that cosine of angle between any two unit vectors is always less than one can be stated as

$$|\cos(\theta)| = |\langle\widehat{\mathbf{x}}, \widehat{\mathbf{y}}\rangle| \leq 1 \tag{29}$$

Moreover, vectors $\mathbf{x}$ and $\mathbf{y}$ are called orthogonal if $(\mathbf{x})^T\ \mathbf{y} = 0$. Orthogonality is probably the most useful concept while working in three dimensional Euclidean space. Inner product spaces and Hilbert spaces generalize these simple geometrical concepts in three dimensional Euclidean space to higher or infinite dimensional vector spaces.

Definition 38 ***(Inner Product Space):*** *An inner product space is a linear vector space X together with an inner product defined on $X \times X$. Corresponding to each pair of vectors $\mathbf{x}, \mathbf{y} \in X$ the inner product $\langle\mathbf{x}, \mathbf{y}\rangle$ of $\mathbf{x}$ and $\mathbf{y}$ is a scalar. The inner product satisfies following axioms.*

1. $\langle\mathbf{x}, \mathbf{y}\rangle = \overline{\langle\mathbf{y}, \mathbf{x}\rangle}$ (complex conjugate)
2. $\langle\mathbf{x} + \mathbf{y}, \mathbf{z}\rangle = \langle\mathbf{x}, \mathbf{z}\rangle + \langle\mathbf{y}, \mathbf{z}\rangle$
3. $\langle\lambda\mathbf{x}, \mathbf{y}\rangle = \overline{\lambda}\,\langle\mathbf{x}, \mathbf{y}\rangle$

   $\langle\mathbf{x}, \lambda\mathbf{y}\rangle = \lambda\,\langle\mathbf{x}, \mathbf{y}\rangle$
4. $\langle\mathbf{x}, \mathbf{x}\rangle \geq 0$ and $\langle\mathbf{x}, \mathbf{x}\rangle = 0$ if and only if $\mathbf{x} = \overline{0}$.

Definition 39 ***(Hilbert Space):*** *A complete inner product space is called as an Hilbert space.*

Here are some examples of commonly used inner product and Hilbert spaces.

Example 40 ***Inner Product Spaces***

1. $X \equiv R^n$ with inner product defined as

$$\langle\mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{30}$$

$$\langle\mathbf{x}, \mathbf{x}\rangle = \sum_{i=1}^{n} (x_i)^2 = \|\mathbf{x}\|_2^2 \tag{31}$$

is a Hilbert space.

2. $X \equiv R^n$ *with inner product defined as*

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y} \tag{32}$$

where W *is a positive definite matrix is a Hilbert space. The corresponding 2-norm is defined as* $\|\mathbf{x}\|_{W,2} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_W} = \sqrt{\mathbf{x}^T W \mathbf{x}}$

3. $X \equiv C^n$ *with inner product defined as*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \overline{x}_i y_i \tag{33}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} \overline{x}_i x_i = \sum_{i=1}^{n} |x_i|^2 = \|\mathbf{x}\|_2^2 \tag{34}$$

is a Hilbert space.

4. *The set of real valued square integrable functions on interval* $[a, b]$ *with inner product defined as*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}(t)\mathbf{y}(t)dt \tag{35}$$

is an Hilbert space and denoted as $L_2[a, b]$. *Well known examples of spaces of this type are the set of continuous functions on* $L_2[-\pi, \pi]$ *or* $L_2[0, 2\pi]$, *which are considered while developing Fourier series expansions of continuous functions on* $[-\pi, \pi]$ *or* $[0, 2\pi]$ *using* $sin(n\pi)$ *and* $cos(n\pi)$ *as basis functions.*

5. *Space of polynomial functions on* $[a, b]$*with inner product*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}(t)\mathbf{y}(t)dt \tag{36}$$

is a inner product space. This is a subspace of $L_2[a, b]$.

6. *Space of complex valued square integrable functions on* $[a, b]$ *with inner product*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \overline{\mathbf{x}}(t)\mathbf{y}(t)dt \tag{37}$$

is an inner product space.

Axioms 2 and 3 imply that the inner product is linear in the first entry. The quantity $\langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ is a candidate function for defining norm on the inner product space. Axioms 1 and 3 imply that $\|\alpha\mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|$ and axiom 4 implies that $\|\mathbf{x}\| > 0$ for $\mathbf{x} \neq \overline{0}$. If we show that $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$satisfies triangle inequality, then $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ defines a norm on space X . We first prove Cauchy-Schwarz inequality, which is generalization of equation (29), and proceed to show that $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ defines the well known 2-norm on X, i.e. $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Lemma 41 ***(Cauchey- Schwarz Inequality):*** *Let X denote an inner product space. For all $\mathbf{x}, \mathbf{y} \in X$,the following inequality holds*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq [\langle \mathbf{x}, \mathbf{x} \rangle]^{1/2} [\langle \mathbf{y}, \mathbf{y} \rangle]^{1/2} \tag{38}$$

The equality holds if and only if $\mathbf{x} = \lambda \mathbf{y}$ or $\mathbf{y} = \overline{0}$

Proof: If $\mathbf{y} = \overline{0}$, the equality holds trivially so we assume $\mathbf{y} \neq \overline{0}$. Then, for all scalars λ,we have

$$0 \leq \langle \mathbf{x} - \lambda\mathbf{y}, \mathbf{x} - \lambda\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \lambda \langle \mathbf{x}, \mathbf{y} \rangle - \overline{\lambda} \langle \mathbf{y}, \mathbf{x} \rangle + |\lambda|^2 \langle \mathbf{y}, \mathbf{y} \rangle \tag{39}$$

In particular, if we choose $\lambda = \dfrac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$, then, using axiom 1 in the definition of inner product, we have

$$\overline{\lambda} = \frac{\overline{\langle \mathbf{y}, \mathbf{x} \rangle}}{\langle \mathbf{y}, \mathbf{y} \rangle} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \tag{40}$$

$$\Rightarrow \quad -\lambda \langle \mathbf{x}, \mathbf{y} \rangle - \overline{\lambda} \langle \mathbf{y}, \mathbf{x} \rangle = -\frac{2 \langle \mathbf{x}, \mathbf{y} \rangle \ \langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \tag{41}$$

$$= \quad -\frac{2 \langle \mathbf{x}, \mathbf{y} \rangle \overline{\langle \mathbf{x}, \mathbf{y} \rangle}}{\langle \mathbf{y}, \mathbf{y} \rangle} = -\frac{2 |\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle} \tag{42}$$

$$\Rightarrow 0 \leq \langle \mathbf{x}, \mathbf{x} \rangle - \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle} \tag{43}$$

$$\text{or} \quad | \langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}$$

The triangle inequality can be can be established easily using the Cauchy-Schwarz inequality as follows

$$\begin{aligned}
\langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle . && (44)\\
&\leq \langle \mathbf{x}, \mathbf{x} \rangle + 2 |\langle \mathbf{x}, \mathbf{y} \rangle| + \langle \mathbf{y}, \mathbf{y} \rangle && (45)\\
&\leq \langle \mathbf{x}, \mathbf{x} \rangle + 2\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} + \langle \mathbf{y}, \mathbf{y} \rangle && (46)
\end{aligned}$$

$$\sqrt{\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle} \leq \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} + \sqrt{\langle \mathbf{y}, \mathbf{y}\rangle} \tag{47}$$

Thus, the candidate function $\sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$ satisfies all the properties necessary to define a norm, i.e.

$$\begin{aligned}
\sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} &\geq 0 \;\forall\; \mathbf{x} \in X \text{ and } \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} = 0 \;\; iff \;\; \mathbf{x} = \overline{0} && (48)\\
\sqrt{\langle \alpha\mathbf{x}, \alpha\mathbf{x}\rangle} &= |\alpha| \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} && (49)\\
\sqrt{\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle} &\leq \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} + \sqrt{\langle \mathbf{y}, \mathbf{y}\rangle} \qquad \text{(Triangle inequality)} && (50)
\end{aligned}$$

Thus, the function $\sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$ indeed defines a norm on the inner product space X. In fact the inner product defines the well known 2-norm on X, i.e.

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} \tag{51}$$

and the triangle inequality can be stated as

$$\|\mathbf{x}+\mathbf{y}\|_2^2 \leq \|\mathbf{x}\|_2^2 + 2\,\|\mathbf{x}\|_2 . \|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2 . = [\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2]^2 \tag{52}$$

$$\text{or } \; \|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2 \tag{53}$$

Definition 42 *(**Angle**) The angle θ between any two vectors in an inner product space is defined by*

$$\theta = \cos^{-1}\left[\frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2}\right] \tag{54}$$

Definition 43 *(**Orthogonal Vectors**): In a inner product space X two vector $\mathbf{x}, \mathbf{y} \in X$ are said to be orthogonal if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$. We symbolize this by $\mathbf{x} \perp \mathbf{y}$. A vector $\mathbf{x}$ is said to be orthogonal to a set S (written as $\mathbf{x} \perp S$) if $\mathbf{x} \perp \mathbf{z}$ for each $\mathbf{z} \in S$.*

Just as orthogonality has many consequences in three dimensional geometry, it has many implications in any inner-product / Hilbert space [2]. The Pythagoras theorem, which is probably the most important result the plane geometry, is true in any inner product space.

Lemma 44 *If $\mathbf{x} \perp \mathbf{y}$ in an inner product space then $\|\mathbf{x}+\mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2$.*

Proof: $\|\mathbf{x}+\mathbf{y}\|_2^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle$.

Definition 45 *(**Orthogonal Set**): A set of vectors S in an inner product space X is said to be an orthogonal set if $\mathbf{x} \perp \mathbf{y}$ for each $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{x} \neq \mathbf{y}$. The set is said to be orthonormal if, in addition each vector in the set has norm equal to unity.*

Note that an orthogonal set of nonzero vectors is linearly independent set. We often prefer to work with an orthonormal basis as any vector can be uniquely represented in terms of components along the orthonormal directions. Common examples of such orthonormal basis are (a) unit vectors along coordinate directions in R^n (b) function $\{sin(nt) : n = 1, 2, ...\}$ and $\{cos(nt) : n = 1, 2, ...\}$ in $L_2[0, 2\pi]$.

Example 46 *Show that function $\langle \mathbf{x}, \mathbf{y} \rangle_W : R^n \times R^n \to R$ defined as*

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y}$$

defines an inner product on when W is a symmetric positive definite matrix.

Solution: *For $\langle \mathbf{x}, \mathbf{y} \rangle_W = x^T W y$ to qualify as inner product, it must satisfy the following all four axioms in the definition of the inner product. We have,*

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y} \quad and \quad \langle \mathbf{y}, \mathbf{x} \rangle_W = \mathbf{y}^T W \mathbf{x}$$

Since W is symmetric, i.e.

$$W^T = W, \ \left[\mathbf{x}^T W \mathbf{y}\right]^T = \mathbf{y}^T W^T \mathbf{x} = \mathbf{y}^T W \mathbf{x}$$

Thus, axiom A1 holds for any $x, y \in R^n$.

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle_W = (\mathbf{x} + \mathbf{y})^T W \mathbf{z} = \mathbf{x}^T W \mathbf{z} + \mathbf{x}^T W \mathbf{z} = \langle \mathbf{x}, \mathbf{z} \rangle_W + \langle \mathbf{y}, \mathbf{z} \rangle_W$$

Thus, axiom A2 holds for any $x, y, z \in R^n$.

$$\begin{aligned} \langle \lambda \mathbf{x}, \mathbf{y} \rangle &= (\lambda \mathbf{x})^T W \mathbf{y} = \lambda (\mathbf{x}^T W \mathbf{y}) = \lambda \langle \mathbf{x}, \mathbf{y} \rangle \\ \langle \mathbf{x}, \lambda \mathbf{y} \rangle &= \mathbf{x}^T W (\lambda \mathbf{y}) = \lambda (\mathbf{x}^T W \mathbf{y}) = \lambda \langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

Thus, axiom A3 holds for any $\mathbf{x}, \mathbf{y} \in R^n$. Since W is positive definite, it follows that $\langle \mathbf{x}, \mathbf{x} \rangle_W = \mathbf{x}^T W \mathbf{x} > 0$ if $x \neq \overline{\mathbf{0}}$ and $\langle \mathbf{x}, \mathbf{x} \rangle_W = \mathbf{x}^T W \mathbf{x} = 0$ if $x = \overline{\mathbf{0}}$. Thus, axiom A4 holds for any $\mathbf{x} \in R^n$. Since all four axioms are satisfied, $\langle \mathbf{y}, \mathbf{x} \rangle_W = \mathbf{y}^T W \mathbf{x}$ is a valid definition of an inner product.

Example 47 *The triangle inequality asserts that, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ belonging to an inner product space*

$$\|\mathbf{x} + \mathbf{y}\|_2 \leq ||\mathbf{y}||_2 + ||\mathbf{x}||_2$$

Does the Cauchy-Schwartz inequality follow from the triangle inequality? Under what condition Schwartz inequality becomes an equality?

Solution: *Squaring both the sides, we have*

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \leq [||\mathbf{y}||_2 + ||\mathbf{x}||_2]^2$$

$$\langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle + 2\langle \mathbf{x},\mathbf{y}\rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2||\mathbf{x}||_2$$

$$||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2\langle \mathbf{x},\mathbf{y}\rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2||\mathbf{x}||_2$$

Since, $||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 \geq 0$ *for any* $\mathbf{x},\mathbf{y} \in X$*, the above inequality reduces to*

$$\langle \mathbf{x},\mathbf{y}\rangle \leq ||\mathbf{y}||_2||\mathbf{x}||_2 \tag{55}$$

The triangle inequality also implies that

$$\|\mathbf{x}-\mathbf{y}\|_2^2 = \langle \mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}\rangle \leq [||\mathbf{y}||_2 + ||\mathbf{x}||_2]^2$$

$$\langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle - 2\langle \mathbf{x},\mathbf{y}\rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2||\mathbf{x}||_2$$

$$||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 - 2\langle \mathbf{x},\mathbf{y}\rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2||\mathbf{x}||_2$$

Since, $||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 \geq 0$ *for any* $\mathbf{x},\mathbf{y} \in X$*, the above inequality reduces to*

$$-\langle \mathbf{x},\mathbf{y}\rangle \leq ||\mathbf{y}||_2||\mathbf{x}||_2$$

i.e.

$$-||\mathbf{y}||_2||\mathbf{x}||_2 \leq \langle \mathbf{x},\mathbf{y}\rangle \tag{56}$$

Combining inequalities (55) and (56), we arrive at the Cauchy-Schwartz inequality

$$-||\mathbf{y}||_2||\mathbf{x}||_2 \leq \langle \mathbf{x},\mathbf{y}\rangle \leq ||\mathbf{y}||_2||\mathbf{x}||_2 \tag{57}$$

i.e.

$$|\langle \mathbf{x},\mathbf{y}\rangle| \leq ||\mathbf{y}||_2||\mathbf{x}||_2 \tag{58}$$

The Cauchy-Schwartz inequality reduces to equality when $\mathbf{y} = \alpha\mathbf{x}$.

5 Gram-Schmidt Process and Orthogonal Polynomials

Given any linearly independent set in an inner product space, it is possible to construct an orthonormal set. This procedure is called Gram-Schmidt procedure. Consider a linearly independent set of vectors $\left\{\mathbf{x}^{(i)}; i = 1,2,3.....n\right\}$ in a inner product space we define $\mathbf{e}^{(1)}$ as

$$\mathbf{e}^{(1)} = \frac{\mathbf{x}^{(1)}}{\|\mathbf{x}^{(1)}\|_2} \tag{59}$$

We form unit vector $\mathbf{e}^{(2)}$ in two steps.

$$\mathbf{z}^{(2)} = \mathbf{x}^{(2)} - \left\langle \mathbf{x}^{(2)},\mathbf{e}^{(1)}\right\rangle \mathbf{e}^{(1)} \tag{60}$$

where$\left\langle \mathbf{x}^{(2)}, \mathbf{e}^{(1)} \right\rangle$ is component of $\mathbf{x}^{(2)}$ along $\mathbf{e}^{(1)}$.

$$\mathbf{e}^{(2)} = \frac{\mathbf{z}^{(2)}}{\left\|\mathbf{z}^{(2)}\right\|_2} \tag{61}$$

.By direct calculation it can be verified that $\mathbf{e}^{(1)} \perp \mathbf{e}^{(2)}$. The remaining orthonormal vectors $\mathbf{e}^{(i)}$ are defined by induction. The vector $\mathbf{z}^{(k)}$ is formed according to the equation

$$\mathbf{z}^{(k)} = \mathbf{x}^{(k)} - \sum_{i=1}^{k-1} \left\langle \mathbf{x}^{(k)}, \mathbf{e}^{(i)} \right\rangle .\mathbf{e}^{(i)} \tag{62}$$

and

$$\mathbf{e}^{(k)} = \frac{\mathbf{z}^{(k)}}{\left\|\mathbf{z}^{(k)}\right\|_2} \quad ; \qquad k = 1, 2, ...n \tag{63}$$

It can be verified by direct computation that $\mathbf{z}^{(k)} \perp \mathbf{e}^{(j)}$ for all $j < k$ as follows

$$\begin{aligned} \left\langle \mathbf{z}^{(k)}, \mathbf{e}^{(j)} \right\rangle &= \left\langle \mathbf{x}^{(k)}, \mathbf{e}^{(j)} \right\rangle - \sum_{i=1}^{k-1} \left\langle \mathbf{x}^{(k)}, \mathbf{e}^{(i)} \right\rangle . \left\langle \mathbf{e}^{(i)}, \mathbf{e}^{(j)} \right\rangle && (64) \\ &= \left\langle \mathbf{x}^{(k)}, \mathbf{e}^{(j)} \right\rangle - \left\langle \mathbf{x}^{(k)}, \mathbf{e}^{(j)} \right\rangle = 0 && (65) \end{aligned}$$

Example 48 ***Gram-Schmidt Procedure in R^3*** : *Consider $X = R^3$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$. Given a set of three linearly independent vectors in R^3*

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \ \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \ \mathbf{x}^{(3)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \tag{66}$$

we want to construct and orthonormal set. Applying Gram Schmidt procedure,

$$\mathbf{e}^{(1)} = \frac{\mathbf{x}^{(1)}}{\left\|\mathbf{x}^{(1)}\right\|_2} . = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{67}$$

$$\begin{aligned} \mathbf{z}^{(2)} &= \mathbf{x}^{(2)} - \left\langle \mathbf{x}^{(2)}, \mathbf{e}^{(1)} \right\rangle .\mathbf{e}^{(1)} && (68) \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \end{aligned}$$

$$\mathbf{e}^{(2)} = \frac{\mathbf{z}^{(2)}}{\left\|\mathbf{z}^{(2)}\right\|_2} . = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{69}$$

$$\mathbf{z}^{(3)} = \mathbf{x}^{(3)} - \left\langle \mathbf{x}^{(3)}, \mathbf{e}^{(1)} \right\rangle .\mathbf{e}^{(1)} - \left\langle \mathbf{x}^{(3)}, \mathbf{e}^{(2)} \right\rangle .\mathbf{e}^{(2)}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{70}$$

$$\mathbf{e}^{(3)} = \frac{\mathbf{z}^{(3)}}{\left\| \mathbf{z}^{(3)} \right\|_2}. = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

Note that the vectors in the orthonormal set will depend on the definition of inner product. Suppose we define the inner product as follows

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y} \tag{71}$$

$$W = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

where W is a positive definite matrix. Then, length of $\left\| \mathbf{x}^{(1)} \right\|_{W,2} = \sqrt{6}$ and the unit vector $\widehat{\mathbf{e}}^{(1)}$ becomes

$$\widehat{\mathbf{e}}^{(1)} = \frac{\mathbf{x}^{(1)}}{\left\| \mathbf{x}^{(1)} \right\|_{W,2}}. = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \end{bmatrix} \tag{72}$$

The remaining two orthonormal vectors have to be computed using the inner product defined by equation 71.

Example 49 ***Gram-Schmidt Procedure in C[a,b]***: *Let X represent set of continuous functions on interval $-1 \leq t \leq 1$ with inner product defined as*

$$\langle \mathbf{x}(t), \mathbf{y}(t) \rangle = \int_{-1}^{1} \mathbf{x}(t) \mathbf{y}(t) dt \tag{73}$$

Given a set of four linearly independent vectors

$$\mathbf{x}^{(1)}(t) = 1; \quad \mathbf{x}^{(2)}(t) = t; \quad \mathbf{x}^{(3)}(t) = t^2; \quad \mathbf{x}^{(4)}(t) = t^3 \tag{74}$$

we intend to generate an orthonormal set. Applying Gram-Schmidt procedure

$$\mathbf{e}^{(1)}(t) = \frac{\mathbf{x}^{(1)}(t)}{\left\| \mathbf{x}^{(1)}(t) \right\|} = \frac{1}{\sqrt{2}} \tag{75}$$

$$\left\langle \mathbf{e}^{(1)}(t), \mathbf{x}^{(2)}(t) \right\rangle = \int_{-1}^{1} \frac{t}{2} dt = 0 \tag{76}$$

$$\mathbf{z}^{(2)}(t) = t - \left\langle \mathbf{x}^{(2)}, \mathbf{e}^{(1)} \right\rangle .\mathbf{e}^{(1)} = t = \mathbf{x}^{(2)}(t) \tag{77}$$

$$\mathbf{e}^{(2)} = \frac{\mathbf{z}^{(2)}}{\|\mathbf{z}^{(2)}\|} \tag{78}$$

$$\left\|\mathbf{z}^{(2)}(t)\right\|^2 = \int_{-1}^{1} t^2 dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2}{3} \tag{79}$$

$$\left\|\mathbf{z}^{(2)}(t)\right\| = \sqrt{\frac{2}{3}} \tag{80}$$

$$\mathbf{e}^{(2)}(t) = \sqrt{\frac{3}{2}}.t \tag{81}$$

$$\begin{aligned}
\mathbf{z}^{(3)}(t) &= \mathbf{x}^{(3)}(t) - \left\langle \mathbf{x}^{(3)}(t), \mathbf{e}^{(1)}(t) \right\rangle .\mathbf{e}^{(1)}(t) - \left\langle \mathbf{x}^{(3)}(t), \mathbf{e}^{(2)}(t) \right\rangle .\mathbf{e}^{(2)}(t) \\
&= t^2 - \frac{1}{2}\left(\int_{-1}^{1} t^2 dt\right)\mathbf{e}^{(1)}(t) - \left(\sqrt{\frac{3}{2}}\int_{-1}^{1} t^3 dt\right)\mathbf{e}^{(2)}(t) \\
&= t^2 - \frac{1}{3} - 0 = t^2 - \frac{1}{3} \qquad (82)
\end{aligned}$$

$$\mathbf{e}^{(3)}(t) = \frac{\mathbf{z}^{(3)}(t)}{\|\mathbf{z}^{(3)}(t)\|} \tag{83}$$

$$\begin{aligned}
\textit{where } \left\|\mathbf{z}^{(3)}(t)\right\|^2 &= \left\langle \mathbf{z}^{(3)}(t), \mathbf{z}^{(3)}(t) \right\rangle = \int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 dt \qquad (84) \\
&= \int_{-1}^{1} \left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right) dt = \left[\frac{t^5}{5} - \frac{2t^3}{9} + \frac{t}{9}\right]_{-1}^{1} \\
&= \frac{2}{3} - \frac{4}{9} + \frac{2}{9} = \frac{18-10}{45} = \frac{8}{45}
\end{aligned}$$

$$\left\|\mathbf{z}^{(3)}(t)\right\| = \sqrt{\frac{8}{45}} = \frac{2}{3}\sqrt{\frac{2}{5}} \tag{85}$$

The orthonormal polynomials constructed above are well known Legandre polynomials. It turns out that

$$\mathbf{e}_n(t) = \sqrt{\frac{2n+1}{2}} p_n(t) \quad ; \quad (n = 0, 1, 2.......) \tag{86}$$

where

$$P_n(t) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dt^n} \left\{ \left(1 - t^2\right)^n \right\} \tag{87}$$

are Legandre polynomials. It can be shown that this set of polynomials forms a orthonormal basis for the set of continuous functions on [-1,1]. First few elements in this orthogonal set are as follows

$$\begin{aligned} P_0(t) &= 1, \quad P_1(t) = t, \quad P_2(t) = \frac{1}{2}(3t^2 - 1), \quad P_3(t) = \frac{1}{2}(5t^3 - 3t) \\ P_4(t) &= \frac{1}{8}(35t^4 - 30t^2 + 3), \quad P_5(t) = \frac{1}{8}(63t^5 - 70t^3 + 15t) \end{aligned}$$

Example 50 ***Gram-Schmidt Procedure in other Spaces***

1. ***Shifted Legandre polynomials:*** $X = C[0,1]$ *and inner product defined as*

$$\langle \mathbf{x}(t), \mathbf{y}(t) \rangle = \int_0^1 \mathbf{x}(t)\mathbf{y}(t)dt \tag{88}$$

   These polynomials are generated starting from linearly independent vectors

$$\mathbf{x}^{(1)}(t) = 1; \quad \mathbf{x}^{(2)}(t) = t; \quad \mathbf{x}^{(3)}(t) = t^2; \quad \mathbf{x}^{(4)}(t) = t^3 \tag{89}$$

   and applying Gram-Schmidt process.

2. ***Hermite Polynomials:*** $X \equiv L^2(-\infty, \infty)$, *i.e. space of continuous functions over* $(-\infty, \infty)$ *with 2 norm defined on it and*

$$\langle \mathbf{x}(t), \mathbf{y}(t) \rangle = \int_{-\infty}^{\infty} \mathbf{x}(t)\mathbf{y}(t)dt \tag{90}$$

   Apply Gram-Schmidt to the following set of vectors in $L^2(-\infty, \infty)$

$$\mathbf{x}^{(1)}(t) = \exp(-\frac{t^2}{2}) \; ; \; \mathbf{x}^{(2)}(t) = t\mathbf{x}^{(1)}(t) \; ; \tag{91}$$

$$\mathbf{x}^{(3)}(t) = t^2\mathbf{x}^{(1)}(t) \, ; ......\mathbf{x}^{(k)}(t) = t^{k-1}\mathbf{x}^{(1)}(t) \, ; .... \tag{92}$$

   First few elements in this orthogonal set are as follows

$$\begin{aligned} H_0(t) &= 1, \quad H_1(t) = 2t, \quad H_2(t) = 4t^2 - 2, \quad H_3(t) = 5t^3 - 12t \\ H_4(t) &= 16t^4 - 48t^2 + 12, \quad H_5(t) = 32t^5 - 160t^3 + 120t \end{aligned}$$

3. ***Laguerre Polynomials:*** $X \equiv L^2(0, \infty)$, *i.e. space of continuous functions over* $(0, \infty)$ *with 2 norm defined on it and*

$$\langle \mathbf{x}(t), \mathbf{y}(t) \rangle = \int_0^{\infty} \mathbf{x}(t)\mathbf{y}(t)dt \tag{93}$$

Apply Gram-Schmidt to the following set of vectors in $L^2(0,\infty)$

$$\mathbf{x}^{(1)}(t) = \exp(-\frac{t}{2}) \ ; \ \mathbf{x}^{(2)}(t) = tx^{(1)}(t)\,; \tag{94}$$
$$\mathbf{x}^{(3)}(t) = t^2\mathbf{x}^{(1)}(t)\,; ......\mathbf{x}^{(k)}(t) = t^{k-1}\mathbf{x}^{(1)}(t)\,; .... \tag{95}$$

The first few Laguerre polynomials are as follows

$$L_0(t) = 1 \ ; \ L_1(t) = 1 - t \ ; \ L_2(t) = 1 - 2t + (1/2)t^2$$
$$L_3(t) = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3 \ ; \ L_4(t) = 1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4$$

6 Induced Matrix Norms

We have already mentioned that set of all $m \times n$ matrices with real entries (or complex entries) can be viewed a linear vector space. In this section, we introduce the concept of *induced norm* of a matrix, which plays a vital role in the numerical analysis. A norm of a matrix can be interpreted as *amplification power* of the matrix. To develop a numerical measure for ill conditioning of a matrix, we first have to quantify this *amplification power* of the matrix.

Definition 51 ***(Induced Matrix Norm):*** *The induced norm of a $m \times n$ matrix $\mathbf{A}$ is defined as mapping from $R^m \times R^n \rightarrow R^+$ such that*

$$\|\mathbf{A}\| = \underset{\mathbf{x} \neq \overline{\mathbf{0}}}{Max} \ \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} \tag{96}$$

In other words, $\|\mathbf{A}\|$ bounds the amplification power of the matrix i.e.

$$\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\| \ \text{ for all } \ \mathbf{x} \in R^n, \mathbf{x} \neq \overline{\mathbf{0}} \tag{97}$$

The equality holds for at least one non zero vector $\mathbf{x} \in R^n$. An alternate way of defining matrix norm is as follows

$$\|\mathbf{A}\| = \underset{\|\widehat{\mathbf{x}}\| = 1}{Max} \ \|\mathbf{A}\widehat{\mathbf{x}}\| \tag{98}$$

Defining $\widehat{\mathbf{x}}$ as

$$\widehat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

it is easy to see that these two definitions are equivalent. The following conditions are satisfied for any matrices $\mathbf{A}, B \in R^m \times R^n$

1. $\|\mathbf{A}\| > 0$ if $\mathbf{A} \neq [\mathbf{0}]$ and $\|[\mathbf{0}]\| = 0$

2. $\|\alpha\mathbf{A}\| = |\alpha|.\|\mathbf{A}\|$

3. $\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$

4. $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\ \|\mathbf{B}\|$

The induced norms, i.e. norm of matrix induced by vector norms on R^m and R^n, can be interpreted as maximum **gain** or **amplification factor** of the matrix.

6.1 Computation of 2-norm

Now, consider **2-norm** of a matrix, which can be defined as follows

$$||\mathbf{A}||_2 = \begin{matrix} max \\ \mathbf{x} \neq 0 \end{matrix} \frac{||\mathbf{Ax}||_2}{||\mathbf{x}||_2} \tag{99}$$

Squaring both sides

$$||\mathbf{A}||_2^2 = \begin{matrix} max \\ \mathbf{x} \neq 0 \end{matrix} \frac{(\mathbf{Ax})^T(\mathbf{Ax})}{(\mathbf{x}^T\mathbf{x})} = \begin{matrix} max \\ \mathbf{x} \neq 0 \end{matrix} \frac{\mathbf{x}^T\mathbf{Bx}}{(\mathbf{x}^T\mathbf{x})}$$

where $\mathbf{B} = \mathbf{A}^T\mathbf{A}$ is a symmetric and positive definite matrix. Positive definiteness of matrix $\mathbf{B}$ requires that

$$\mathbf{x}^T\mathbf{Bx} > 0 \; if \; \mathbf{x} \neq \; \overline{0} \quad \text{and} \;\; \mathbf{x}^T\mathbf{Bx} = 0 \text{ if and only if } \; \mathbf{x} = \overline{0} \tag{100}$$

If columns of $\mathbf{A}$ are linearly independent, then it implies that

$$\mathbf{x}^T\mathbf{Bx} = (\mathbf{Ax})^T(\mathbf{Ax}) > 0 \; if \; \mathbf{x} \neq \overline{\mathbf{0}} \tag{101}$$

$$= 0 \qquad if \; \mathbf{x} = \overline{\mathbf{0}} \tag{102}$$

Now, a positive definite symmetric matrix can be diagonalized as

$$\mathbf{B} = \mathbf{\Psi\Lambda\Psi}^T \tag{103}$$

Where Ψ is matrix with eigen vectors as columns and $\mathbf{\Lambda}$ is the diagonal matrix with eigenvalues of $\mathbf{B}\,(=\,\mathbf{A}^T\mathbf{A})$ on the main diagonal. Note that in this case $\mathbf{\Psi}$ is unitary matrix ,i.e.,

$$\mathbf{\Psi\Psi}^T = \mathbf{I} \;\text{ i.e }\; \mathbf{\Psi}^T = \mathbf{\Psi}^{-1} \tag{104}$$

and eigenvectors are orthogonal. Using the fact that $\mathbf{\Psi}$ is unitary, we can write

$$\mathbf{x}^T\mathbf{x} = \mathbf{x}^T\mathbf{\Psi}\mathbf{\Psi}^T\mathbf{x} = \mathbf{y}^T\mathbf{y} \tag{105}$$

This implies that

$$\frac{\mathbf{x}^T\mathbf{B}\mathbf{x}}{(\mathbf{x}^T\mathbf{x})} = \frac{\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}}{(\mathbf{y}^T\mathbf{y})} \tag{106}$$

where $\mathbf{y} = \mathbf{\Psi}^T\mathbf{x}$. Suppose eigenvalues $\lambda_i$of $\mathbf{A}^T\mathbf{A}$ are numbered such that

$$0 < \lambda_1 \leq \lambda_2 \leq ...... \leq \lambda_n \tag{107}$$

Then, we have

$$\frac{\mathbf{y}^T\Lambda\mathbf{y}}{(\mathbf{y}^T\mathbf{y})} = \frac{(\lambda_1\mathbf{y}_1^2 + \lambda_2\mathbf{y}_2^2 + ... + \lambda_n\mathbf{y}_n^2)}{(\mathbf{y}_1^2 + \mathbf{y}_2^2 + ... + \mathbf{y}_n^2)} \leq \lambda_n \tag{108}$$

which implies that

$$\frac{\mathbf{y}^T\Lambda\mathbf{y}}{(\mathbf{y}^T\mathbf{y})} = \frac{\mathbf{x}^T B\mathbf{x}}{(\mathbf{x}^T\mathbf{x})} = \frac{\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x}}{(\mathbf{x}^T\mathbf{x})} \leq \lambda_n \tag{109}$$

The equality holds only at the corresponding eigenvector of $\mathbf{A}^T\mathbf{A}$, i.e.,

$$\frac{\left[\mathbf{v}^{(n)}\right]^T(\mathbf{A}^T\mathbf{A})\mathbf{v}^{(n)}}{\left[\mathbf{v}^{(n)}\right]^T\mathbf{v}^{(n)}} = \frac{\left[\mathbf{v}^{(n)}\right]^T\lambda_n\mathbf{v}^{(n)}}{\left[\mathbf{v}^{(n)}\right]^T\mathbf{v}^{(n)}} = \lambda_n \tag{110}$$

Thus, 2 norm of matrix $\mathbf{A}$ can be computed as follows

$$||\mathbf{A}||_2^2 = \begin{array}{c} max \\ \mathbf{x} \neq 0 \end{array} \; ||\mathbf{A}x||^2/||\mathbf{x}||^2 = \lambda_{\max}(\mathbf{A}^T\mathbf{A}) \tag{111}$$

i.e.

$$||\mathbf{A}||_2 = [\lambda_{max}(\mathbf{A}^T\mathbf{A})]^{1/2} \tag{112}$$

where $\lambda_{max}(\mathbf{A}^T\mathbf{A})$ denotes maximum magnitude eigenvalue or the *spectral radius* of $\mathbf{A}^T\mathbf{A}$.

6.2 Other Matrix Norms

Other commonly used matrix norms are

- **1-norm**: Maximum over column sums

$$||\mathbf{A}||_1 = \begin{array}{c} \max \\ 1 \leq j \leq n \end{array} \; \left[\sum_{i=1}^{n}|a_{ij}|\right] \tag{113}$$

- $\infty-$**norm:** Maximum over row sums

$$||\mathbf{A}||_\infty = \begin{array}{c} \max \\ 1 \leq i \leq n \end{array} \; \left[\sum_{j=1}^{n}|a_{ij}|\right] \tag{114}$$

Remark 52 *There are other matrix norms, such as Frobenious norm, which are not induced matrix norms. Frobenious norm is defined as*

$$||\mathbf{A}||_F = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right]^{1/2}$$

7 Summary

In this chapter, some fundamental concepts from functional analysis have been reviewed. We begin with the concept of a general vector space and define various algebraic and geometric structures like norm and inner product. We then move to define inner product, which generalizes the concept of dot product, and angle between vectors. We also interpret the notion of orthogonality in a general inner product space and develop Gram-Schmidt process, which can generate an orthonormal set from a linearly independent set. Definition of inner product and orthogonality paves the way to generalize the concept of projecting a vector onto any sub-space of an inner product space. In the end, we discuss induced matrix norms, which play an important role in the analysis of numerical schemes.

8 Exercise

1. While solving problems using a digital computer, arithmetic operations can be performed only with a limited precision due to finite word length. Consider the vector space $X \equiv R$ and discuss which of the laws of algebra (associative, distributive, commutative) are not satisfied for the floating point arithmetic in a digital computer.

2. Show that the solution of the differential equation

$$\frac{d^2x}{dt^2} + x = 0$$

   is a linear space. What is the dimension of this space?

3. Show that functions 1, exp(t), exp(2t), exp(3t) are linearly independent over any interval [a,b].

4. Does the set of functions of the form

$$f(t) = 1/(a + bt)$$

   constitute a linear vector space?

5. Give an example of a function which is in $\mathbf{L}_1[0,1]$ but not in $\mathbf{L}_2[0,1]$.

6. Decide linear dependence or independence of

   (a) (1,1,2), (1,2,1), (3,1,1)

   (b) $\left(\mathbf{x}^{(1)}-\mathbf{x}^{(2)}\right), \left(\mathbf{x}^{(2)}-\mathbf{x}^{(3)}\right), \left(\mathbf{x}^{(3)}-\mathbf{x}^{(4)}\right), \left(\mathbf{x}^{(4)}-\mathbf{x}^{(1)}\right)$ for any $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)}$

   (c) (1,1,0), (1,0,0), (0,1,1), (x,y,z) for any scalars x,y,z

7. Describe geometrically the subspaces of R^3 **spanned** by following sets

   (a) (0,0,0), (0,1,0), (0,2,0)

   (b) (0,0,1), (0,1,1), (0,2,0)

   (c) all six of these vectors

   (d) set of all vectors with positive components

8. Consider the space X of all $n \times n$ matrices. Find a basis for this vector space and show that set of all lower triangular $n \times n$ matrices forms a subspace of X.

9. Determine which of the following definitions are valid as definitions for norms in $\mathbf{C}^{(2)}[a,b]$

   (a) $max\,|\mathbf{x}(t)| + \max|\mathbf{x}'(t)|$

   (b) $\max|\mathbf{x}'(t)|$

   (c) $|\mathbf{x}(a)| + \max|\mathbf{x}'(t)|$

   (d) $|\mathbf{x}(a)| \max|\mathbf{x}(t)|$

10. In a normed linear space X the set of all vectors $\mathbf{x} \in X$ such that $\|\mathbf{x}-\overline{\mathbf{x}}\| \leq 1$ is called unit ball centered at $\overline{\mathbf{x}}$.

    (a) Sketch unit balls in R^2 when 1, 2 and ∞ norms are used.

    (b) Sketch unit ball in C[0,1] when maximum norm is used.

    (c) Can you draw picture of unit ball in $\mathrm{L}_2[0,1]$?

11. Two norms $\|.\|_a$ and $\|.\|_b$ are said to be equivalent if there exists two positive constants c_1 and c_2,independent of $\mathbf{x}$, such that

$$c_1 \|\mathbf{x}\|_a \leq \|\mathbf{x}\|_b \leq c_2 \|\mathbf{x}\|_a$$

    Show that in R^n the 2 norm (Euclidean norm) and $\infty-$norm (maximum norm) are equivalent.

12. Show that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$$

13. A norm $\|.\|_a$is said to be stronger than another norm $\|.\|_b$if

$$\begin{matrix} \lim \\ k \rightarrow \infty \end{matrix} \quad \left\|\mathbf{x}^{(k)}\right\|_a = 0 \Rightarrow \begin{matrix} \lim \\ k \rightarrow \infty \end{matrix} \quad \left\|\mathbf{x}^{(k)}\right\|_b = 0$$

but not vice versa. For C[0,1], show that the maximum norm is stronger than 2 norm.

14. Show that function $\|\mathbf{x}\|_{2,W} : R^n \rightarrow R$ defined as

$$\|\mathbf{x}\|_{2,W} = \sqrt{\mathbf{x}^T W \mathbf{x}}$$

defines a norm on when W is a positive definite matrix.

15. Consider $X = R^3$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T W \mathbf{y}$. Given a set of three linearly independent vectors in R^3

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ; \ \mathbf{x}^{(2)} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} ; \ \mathbf{x}^{(3)} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

we want to construct and orthonormal set. Applying Gram Schmidt procedure,

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y}$$

$$W = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

16. **Gram-Schmidt Procedure in C[a,b]**: Let X represent set of continuous functions on interval $0 \leq t \leq 1$ with inner product defined as

$$\langle \mathbf{x}(t), \mathbf{y}(t) \rangle = \int_0^1 w(t)\mathbf{x}(t)\mathbf{y}(t)dt$$

Given a set of four linearly independent vectors

$$\mathbf{x}^{(1)}(t) = 1; \quad \mathbf{x}^{(2)}(t) = t; \quad \mathbf{x}^{(3)}(t) = t^2;$$

find orthonormal set of vectors if (a) $w(t) = 1$ (Shifted Legandre Polynomials) (b) $w(t) = t(1-t)$ (Jacobi polynomials).

17. Show that in C[a,b] with maximum norm, we cannot define an inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ such that $\langle \mathbf{x}, \mathbf{x} \rangle^{1/2} = \|\mathbf{x}\|_\infty$. In other words, show that in $C[a,b]$ the following function

$$\langle f(t), g(t) \rangle = \max_t \; |x(t)y(t)|$$

cannot define an inner product.

18. In $C^{(1)}[a,b]$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}'(t)\mathbf{y}'(t)dt + \mathbf{x}(a)\mathbf{y}(a)$$

an inner product?

19. Show that in $C^{(1)}[a,b]$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b w(t)\mathbf{x}(t)\mathbf{y}(t)dt$$

with $w(t) > 0$ defines an inner product.

20. Show that parallelogram law holds in any inner product space.

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

Does it hold in C[a,b] with maximum norm?

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