Module 3: Approximation in Hypersonic Inviscid Flows

Lecture-7: Hypersonic flow relations

7.1. Shock relations for hypersonic flow

Consider a supersonic flow passing from a compression corner as shown in Fig. 7.1.

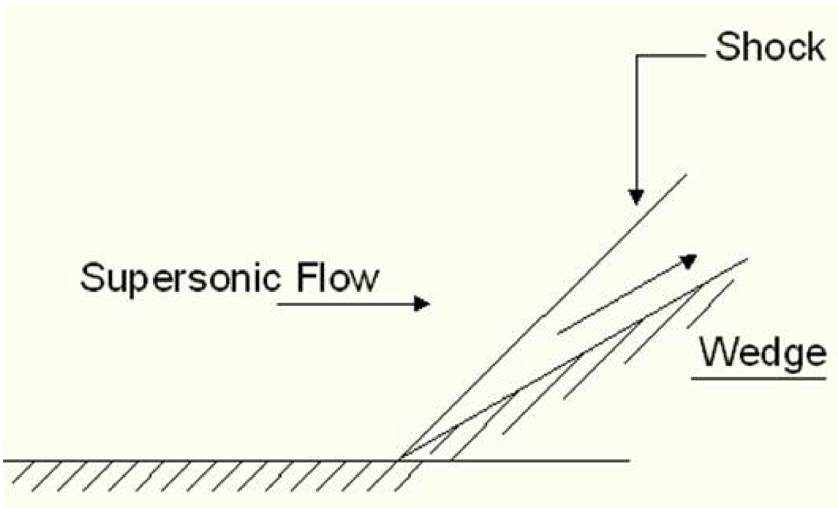

Fig. 7.1 Supersonic flow over a compression corner.

Following are the oblique shock relations for supersonic flow where β is the shock angle.

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 Sin^2(\beta) - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2 \sin^2\beta}{(\gamma-1)M_1^2 \sin^2\beta + 2}$$

Here 1 is the freestream or pre-shock condition while 2 is the post shock condition.

Since hypersonic flows have very high Mach number we can assume that M 1 >>1. This leads to the modification as, $M^2 \sin^2 ß - 1 \approx M^2 \sin^2 ß$. Hence the relation for pressure ratio is,

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1 \sin^2\beta) \qquad (7.1)$$

Similarly density ratio can be rewritten as,

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)}{(\gamma-1)} \qquad (7.2)$$

Ratio for temperature for very high freestream Mach numbers can be then expressed as,

$$\frac{T_2}{T_1} = \frac{p_2/p_1}{\rho_2/\rho_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1{}^2 \sin^2\beta \qquad (7.3)$$

The θ-β-M relation for supersonic flow as

$$\tan(\theta) = 2\cot(\beta)\left[\frac{M_1{}^2\sin^2\beta - 1}{M_1{}^2(\gamma+\cos 2\beta)+2}\right].$$

For small wedge and shock angles we can write $\sin\beta \approx \beta$ and $\cos 2\beta \approx 1$ and replacing M_1 by M we get.

$$\theta = \frac{2}{\beta}\left[\frac{M^2\beta^2 - 1}{M^2(\gamma+1)+2}\right]$$

At the limit of high Mach numbers the above expression reduces to,

$$\theta = \frac{2}{\beta}\left[\frac{\beta^2}{\gamma+1}\right]$$

$$\beta = \frac{\gamma+1}{2}\theta \qquad (7.4)$$

This expression is valid for high Mach numbers and small deflection angles

7.2. Pressure coefficient for hypersonic flow

We know that the pressure coefficient at any point can be calculated as

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{p - p_\infty}{q_\infty} = \frac{p_2 - p_1}{q_1}$$

Here, dynamic pressure = $q_1 = \frac{1}{2}\rho_\infty V_\infty^{\ 2} = \frac{\gamma}{2} M_\infty^{\ 2} p_\infty = \frac{\gamma}{2} M_1^{\ 2} p_1$

Therefore the expression, for pressure coefficient becomes,

$$c_p = \frac{p_2 - p_1}{\frac{\gamma}{2} M_1^{\ 2} p_1} = \frac{2}{\gamma M^2}\left(\frac{p_2}{p_1} - 1\right)$$

$$c_p = \frac{2}{\gamma M^2}\left[\frac{2\gamma}{\gamma + 1}(M_1^{\ 2}\sin^2\beta - 1)\right]$$

$$c_p = \frac{2}{M_1^{\ 2}} \cdot \frac{2}{(\gamma + 1)}(M_1^{\ 2}\sin^2\beta - 1)$$

For high Mach number we have M>>1 we can the get the simplified expression as,

$$c_p = \frac{4}{(\gamma + 1)M_1^{\ 2}} M_1^{\ 2}\sin^2\beta$$

We can express the shock angle using Eq. 7.4. and also assume shock angle to be small ($\sin\beta \approx \beta$). This leads to,

$$c_p = \frac{4}{\gamma + 1}\left[\frac{\gamma + 1}{2}\right]^2 \theta^2 \qquad (7.5)$$

This expression is valid for small flow deflection angles and very high Mach numbers.

Lecture-8: Hypersonic flow relations

8.1. Expansion relation for hypersonic flow

We know that supersonic flow expands if it is provided with the outward turning as shown in Fig. 8.1

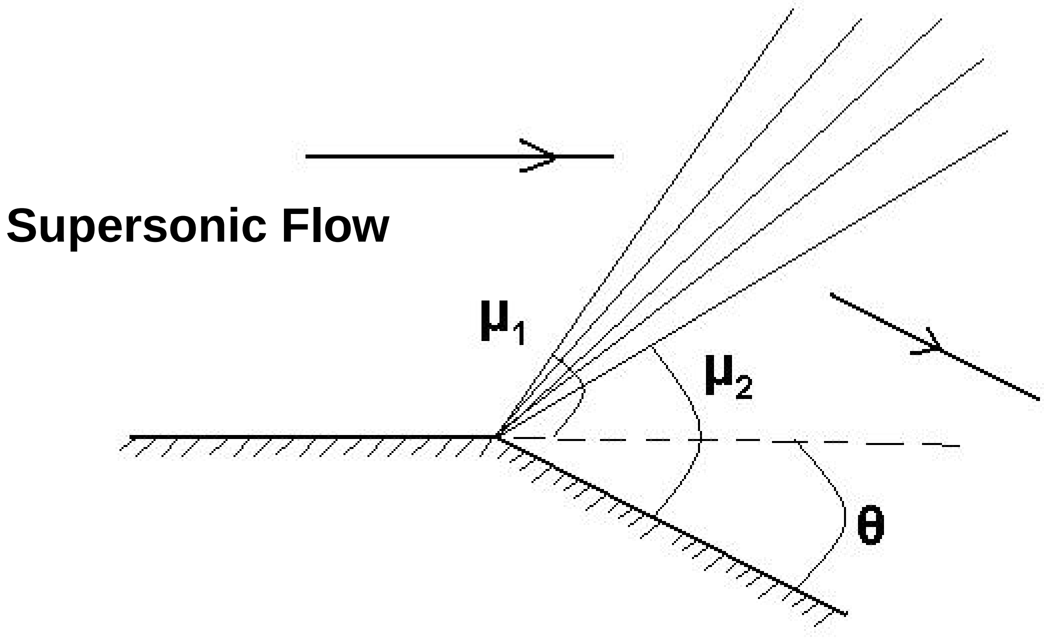

Fig. 8.1. Supersonic flow at an expansion corner.

The expression for the Prandtl-Meyer function can be used to get the post expansion properties of the flow. The expression for this function is as,

$$v(M)=\sqrt{\frac{\gamma+1}{\gamma-1}}[\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}}(M^2-1)]-\tan^{-1}\sqrt{M^2-1} \quad (8.1)$$

If we consider Mach number to be very high (M>>1) then,

$$v(M)=\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}(\sqrt{\frac{\gamma-1}{\gamma+1}}M)-\tan^{-1}(M)$$

Lets consider $x=\sqrt{\dfrac{\gamma+1}{\gamma-1}}$

Therefore,

$$v(M)=x\tan^{-1}(\frac{1}{x}M)-\tan^{-1}(M) \quad (8.2)$$

Or, if we consider $y=\sqrt{\dfrac{\gamma-1}{\gamma+1}}$

Therefore,

$$v(M)=\frac{1}{y}\tan^{-1}(yM)-\tan^{-1}(M) \quad (8.3)$$

But we know that $\tan(y)=\dfrac{\pi}{2}-\tan^{-1}(\dfrac{1}{yM})$

And $\tan^{-1}(1/z)=1/z-1/3z^3+1/5z^5-1/7z^7+1/9z^9-......$

So $\tan^{-1}(y)=\dfrac{\pi}{2}-\tan^{-1}(1/y)=\pi/2-[1/x-1/3x^3+.......]$

Expressing the Eq. 8.3 using this relation we get,

$$v(M)=\frac{1}{y}[\pi/2-(1/yM-1/3(yM^3)+......)]-[\pi/2-(1/M-1/3M^3+.......)]$$

$$v(M)=\pi/2y-1/y(1/yM-1/3(yM)^3+.....)-\pi/2+(1/M-1/3M^3+.....)$$

Neglecting the tems having higher powers of Mach numbers in the denominator, we are left with the first term

$$v(M)=\pi/2y-1/y2M-\pi/2+1/M \quad (8.4)$$

We can use this simple expression of Prandtl-Meyer function for expressing the deflection by the expansion corner for high Mach numbers as,

$$\theta = v(M_2) - v(M_1)$$

$$\theta = \frac{1}{y^2}\left(\frac{1}{M_1} - \frac{1}{M_2}\right) + \left(\frac{1}{M_2} - \frac{1}{M_1}\right)$$

$$y^2 = \frac{\gamma - 1}{\gamma + 1}$$

$$y^2 = \frac{\gamma + 1}{\gamma - 1}$$

$$\theta = \left(\frac{1}{M_1} - \frac{1}{M_2}\right)\left(\frac{1}{y^2} - 1\right) = \left(\frac{1}{M_1} - \frac{1}{M_2}\right)\left(\frac{\gamma + 1}{\gamma - 1} - 1\right)$$

$$\theta = \left(\frac{1}{M_1} - \frac{1}{M_2}\right)\frac{2}{\lambda - 1}$$

$$\theta = \frac{2}{\lambda - 1}\left(\frac{1}{M_1} - \frac{1}{M_2}\right) \tag{8.5}$$

Since total pressure is constant across the expansion fan we can calculate the pressure ratio for a given hypersonic expansion.

$$\frac{p_2}{p_1} = \frac{p_2/p_0}{p_1/p_0} = \frac{\left(1 + \frac{\gamma - 1}{2} M_2^{\,2}\right)^{\frac{-\gamma}{\gamma - 1}}}{\left(1 + \frac{\gamma - 1}{2} M_1^{\,2}\right)^{\frac{-\gamma}{\gamma - 1}}}$$

Where, $\frac{p_0}{p} = \left(1 - \frac{\lambda - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$

$$\frac{p_2}{p_1} = \frac{\left(1 + \frac{\gamma - 1}{2} M_1^{\,2}\right)^{\frac{\gamma}{\gamma - 1}}}{\left(1 + \frac{\gamma - 1}{2} M_2^{\,2}\right)^{\frac{\gamma}{\gamma - 1}}}$$

Since M>>1

$$\frac{p_2}{p_1} \approx \left(\frac{M_1}{M_2}\right)^{\frac{\gamma}{\gamma - 1}} \tag{8.6}$$

8.3. Some other hypersonic flow relations

Using θ we can represent M_2 in terms of M_1

$$\theta = \frac{2}{\gamma-1}\left[\frac{1}{M_1} - \frac{1}{M_2} \right]$$

$$M_1\theta = \frac{2M_1}{\gamma-1}\left[\frac{1}{M_1} - \frac{1}{M_2} \right]$$

$$M_1\theta = \frac{2}{\gamma-1}\left[1 - \frac{M_1}{M_2} \right]$$

$$\frac{\gamma-1}{2}M_1\theta = 1 - \frac{M_1}{M_2}$$

$$\frac{M_1}{M_2} = 1 - \frac{\gamma-1}{2}M_1\theta$$

But we have seen in Eq. 8.6 that pressure ratio is proportional to the Mach number ratio. Hence we can express the pressure ratio for expansion corner in terms of deflection angle and upstream Mach number as,

$$\frac{p_2}{p_1} = \left[1 - \frac{\gamma-1}{2}M_1\theta\right]^{2\gamma/\gamma-1} \tag{8.7}$$

It will be seen later the importance of $M_1\theta$ as the similarity parameter in hypersonic flow, moreover let's represented it as k.

$$\frac{p_2}{p_1} = \left[1 - \frac{\gamma-1}{2}k\right]^{2\gamma/\gamma-1}$$

We also know that the θ-β-M relation for supersonic flow is as

$$\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2}\right]$$

For small θ we have β also small and also for large Mach numbers we have M>>1

Therefore this relation leads to,

$$M_1^2\beta^2 - 1 = \beta\theta\left[\frac{\gamma+1}{2}M_1^2\right]$$

Dividing by M $^2\theta^2$

$$(\frac{\beta}{\theta})^2-\frac{1}{M^2\theta^2}=\frac{\gamma+1}{2}(\frac{\beta}{\theta})$$

$$(\frac{\beta}{\theta})^2-\frac{\gamma+1}{2}(\frac{\beta}{\theta})-\frac{1}{M^2\theta^2}=0 \qquad (8.7)$$

Hence,

$$\frac{\beta}{\theta}=\frac{\gamma+1}{4}+\sqrt{(\frac{\gamma+1}{4})^2+\frac{1}{M^2\theta^2}} \qquad (8.8)$$

Here also we can see that β/θ is represented as the function of hypersonic similarity parameter k. We have neglected other root since non physical.

$$\frac{\beta}{\theta}=\frac{\gamma+1}{4}+\sqrt{(\frac{\gamma+1}{4})^2+\frac{1}{k^2}} \qquad (8.9)$$

We can also obtain the expressions across shock in terms of β/θ relation & k. Lets consider the expression for pressure ratio across the shock, where M_1 =M is the Mach number ahead of the shock

$$\frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}(M^2\sin^2\beta-1)=1+\frac{2\gamma}{\gamma+1}(M^2\beta^2-1) \qquad (8.10)$$

From Eq. 8.7 we have,

$$\left(\frac{\beta}{\theta}\right)^2-\frac{\gamma+1}{2}[\frac{\gamma+1}{4}+\sqrt{(\frac{\gamma+1}{4})^2+\frac{1}{M^2\theta^2}}]-\frac{1}{M^2\theta^2}=0$$

$$\left(\frac{\beta}{\theta}\right)^2=\frac{\gamma+1}{2}[\frac{\gamma+1}{4}+\sqrt{(\frac{\gamma+1}{4})^2+\frac{1}{M^2\theta^2}}]+\frac{1}{M^2\theta^2}$$

$$\beta^2=\{\frac{\gamma+1}{2}[\frac{\gamma+1}{4}+\sqrt{(\frac{\gamma+1}{4})^2+\frac{1}{M^2\theta^2}}]+\frac{1}{M^2\theta^2}\}\theta^2$$

Putting this expression in Eq. 8.10 we get,

$$\frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}\left\{\left[M^2\left\{\frac{\gamma+1}{2}\left[\frac{\gamma+1}{4}+\sqrt{\left(\frac{\gamma+1}{4}\right)^2+1/M^2\theta^2}\right]+1/M^2\theta^2\right\}\theta^2\right]-1\right\}$$

$$\frac{p_2}{p_1}=1+\left\{\left[\gamma M^2\theta^2\left[\frac{\gamma+1}{4}+\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{M^2\theta^2}}\right]+M^2\theta^2\frac{1}{M^2\theta^2}\frac{2\gamma}{\gamma+1}\right]-\frac{2\gamma}{\gamma+1}\right\}$$

$$\frac{p_2}{p_1}=1+\left\{\gamma M^2\theta^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{M^2\theta^2}}\right\}+\frac{\gamma(\gamma+1)}{4}M^2\theta^2+\frac{2\gamma}{\gamma+1}-\frac{2\gamma}{\gamma+1}$$

$$\frac{p_2}{p_1}=1+\gamma M^2\theta^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{M^2\theta^2}}+\frac{\gamma(\gamma+1)}{4}M^2\theta^2$$

However,

$$M\theta=k$$

$$\frac{p_2}{p_1}=1+\gamma k^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{k^2}}+\frac{k^2\gamma(\gamma+1)}{4} \tag{8.11}$$

Now we can obtain the expression for pressure coefficient using this relation as,

$$c_p=\frac{p_2-p_1}{q_1}\ \&\ q_1=\frac{\gamma_1}{2}M_1^{\,2}p_1$$

$$c_p=\frac{p_2-p_1}{\frac{\gamma}{2}M_1^{\,2}p_1}=\frac{2}{\gamma M^2}\left(\frac{p_2}{p_1}-1\right)$$

$$c_p=\frac{2}{\gamma M^2}\left[1+\gamma k^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{k^2}}+\frac{\gamma(\gamma+1)}{4}k^2-1\right]$$

$$c_p=\frac{2}{\gamma M^2}\frac{\theta^2}{\theta^2}[\gamma k^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{k^2}}+\frac{\gamma(\gamma+1)}{4}k^2]$$

$$c_p=\frac{2\theta^2}{\gamma k^2}\left[\gamma k^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{k^2}}+\gamma k^2\left(\frac{\gamma+1}{4}\right)\right]$$

$$c_p=2\theta^2\left[\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{k^2}}+\left(\frac{\gamma+1}{4}\right)\right] \tag{8.12}$$

Therefore any hypersonic relation can be obtained using $M_1 >> 1$ condition. We can also obtain the relations in terms of k=Mθ

Lecture-9: Local Surface Inclination techniques (Newtonian Technique)

9.1. Introduction

Hypersonic flows are nonlinear by nature. Making Mach number tends to infinity cannot lead to any linear theory for small deflections in hypersonic region unlike in supersonic case to evaluate the flow variables. But there are approximate local inclination methods for hypersonic flowfield predictions (mainly pressure) where we need not have to solve for whole flow field.

9.1.1 Newtonian Technique or 'Sine-squared' law

Newton proposed that force on any object offered by fluid flowing, is proportional with square of sine angle of the flow deflection. However this proposition was made by Newton for low speed flows but it is highly appreciated for hypersonic situations. Consider a plate in the flowfield as shown in Fig. 8.1.

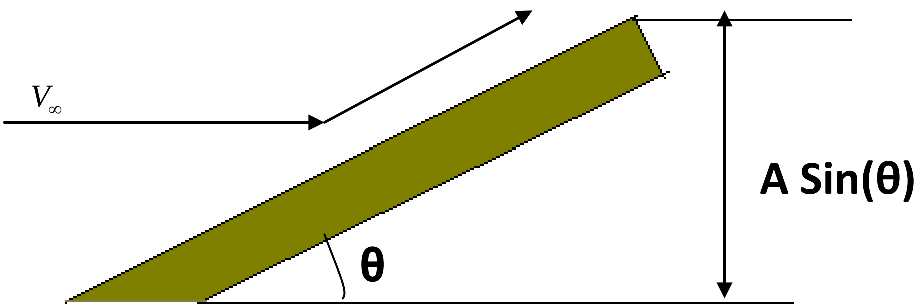

Fig. 8.1 Schematic of the hypersonic flow over flat plate.

Here fluid particle looses normal momentum and conserves tangential momentum. If V_∞ is velocity of fluid approaching an inclined surface with angle θ then

Velocity normal to the surface = $V_\infty \sin(\theta)$

Mass flux blocked by surface = $\rho_\infty V_\infty \sin(\theta)$

Momentum lost by fluid per unit time= $\rho_\infty V_\infty A \sin(\theta)\{V_\infty \sin(\theta)\}$

$$= \quad \rho_\infty V_\infty{}^2 A \sin^2(\theta)$$

This lost momentum is converted in to the forced getting applied by fluid. Hence,

$$F = \rho_\infty V_\infty{}^2 A \sin^2(\theta)$$

$$\frac{F}{A} = \rho_\infty V_\infty{}^2 \sin^2(\theta)$$

Actually Newton had assumed this force is due to loss of momentum of individual particle. Therefore no interaction among fluid particles has been considered. However this force is the pressure force which is due to the random motion of particles.

$$p - p_\infty = \rho_\infty V_\infty{}^2 \sin^2(\theta)$$

The reference pressure is subtracted from the applied pressure since pressure always act in difference while getting applied as a force. Here p_∞ is considered to be the pressure of freestream acting on back side of plate. Hence,

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty v_\infty{}^2} = 2\sin^2\theta$$

$$c_p = 2\sin^2\theta \qquad (9.1)$$

This equation clearly demonstrates the fact that the pressure coefficient at a point is proportional to the sin square of the flow deflection angle. At large Mach numbers and moderately small deflection angles, shock angle is equal to the deflection angle where flow particle eventually hits the surface without any prior warning or deflection. This situation matches well with the Newton's sine squared law. This is the reason for this law to be appreciated for hypersonic situations.

We can use flow Eq. 9.1 for estimation on force acting on any hypersonic configuration. However the flow deflection angle between should be known for this estimation. For the surface which is in shadow Cp should be taken as zero. Using this principle we can calculate the lift to drag ratio for the flat plate configuration shown in Fig. 9.1.

$$c_d = \frac{drag}{\frac{1}{2}\rho_\infty v_\infty{}^2 A} = c_p \sin\theta$$

$$c_d = 2\sin^2\theta\sin\theta = 3\sin^3\theta$$

$$c_l = \frac{lift}{\frac{1}{2}\rho_\infty v_\infty{}^2 A} = c_p \cos\theta$$

$$c_l = 2\sin^2\theta\cos\theta$$

$$\frac{c_l}{c_d} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \frac{L}{D}$$

This result is applicable to inviscid supersonic or hypersonic flow over flat plate

Major depictions from this flow over flat plate are:

1) L/D increases as θ decreases and becomes ∞ when θ is 0. This is logical if flow is inviscid where we are not having any shear or skin friction drag. However this is incorrect in case of viscous flow
2) C_L increases with increase in θ and becomes maximum at around $\theta \approx 55$ which is almost the practical condition.
3) For small θ (below 15 degree) we have nonlinear variation of C_l which is unlikely for subsonic and supersonic flows.

For flow over cylinder & sphere we get similar results. From Newtonian theory or sine squared law,

For sphere $c_d = 1 \rightarrow (c_l = 0)$

For cylinder $c_d = 4/3 \rightarrow (c_l = 0)$

These results of sphere, cylinder and plate are independent of Mach number where Mach number does not appear explicitly. Hence these results demonstrate the validity of famous Mach number independence principle which will be discussed in later chapter.

Lecture-10: Local Surface Inclination techniques (Modified Newtonian Technique)

10.1. Modified Newtonian theory

Modification to the above mentioned Newtonian theory has been proposed by Laster Lees where,

$$c_p = c_{p\max} \sin^2\theta = c_{p_{o_2}} \sin^2\theta \qquad (10.1)$$

$$c_{p_o} = \frac{P_{02} - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$$

Since
$$c_{p_o} = \frac{P_{02} - p_\infty}{\frac{\gamma}{2} P_\infty M_\infty^2} = \frac{2}{\gamma M_\infty{}^2}\left[\frac{p_{0_2}}{p_\infty} - 1\right]$$

Here all the properties, upstream of the shock with subscript 1 are replaces by freestream subscript.

Since Rayleigh Pitot tube formula is

$$\frac{p_{0_2}}{p_\infty} = \left[\frac{(\gamma+1)^2 M_\infty{}^2}{4\gamma M_\infty{}^2 - 2(\gamma-1)}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{1-\gamma+2\gamma M_\infty{}^2}{\gamma+1}\right]$$

$$c_{p_0} = \left[\frac{M_\infty{}^2(\gamma+1)^2}{M_\infty{}^2\left(4\gamma - \frac{2(\gamma+1)}{M_\infty{}^2}\right)}\right]^{\gamma/\gamma-1}\left[\frac{\frac{(1-\lambda)}{M_\infty{}^2} + 2\gamma}{(\gamma+1)}\right] M_\infty{}^2$$

But we know that,

$$c_p = c_{p_0}\sin^2\theta$$

$$c_p = \left\{\left[\frac{(\gamma+1)^2}{4\gamma}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}\right]M_\infty^{\ 2}-1\right\}\frac{2}{\gamma M_\infty^{\ 2}}\sin^2\theta$$

$$c_p = \left\{\left[\frac{(\gamma+1)^2}{4\gamma}\right]^{\frac{\gamma}{\gamma-1}}\frac{4}{\gamma+1}-\frac{2}{\gamma M_\infty^{\ 2}}\right\}\sin^2\theta \qquad (10.2)$$

For high Mach numbers the above expression reduces to,

$$c_p = \left\{\left[\frac{(\gamma+1)^2}{4\gamma}\right]^{\frac{\gamma}{\gamma-1}}\frac{4}{\gamma+1}\right\}\sin^2\theta$$

Coefficient of $\sin^2\theta$ is 1.839 for $\gamma = 1.4$ and 2 for $\gamma = 1$. However in practical situations as well when M_∞ becomes very large γ tends to 1 which in turn results in the expected Newtonian theory arrangement. Results of modified Newtonian theory holds good for blunt body configuration, since this theory predicts exact pressure at stagnation point.

10.2. Newtonian-Busemann theory

This theory modifies basic Newtonian theory with implementation of centrifugal correction. The force or pressure predicted by Newtonian theory is appropriate for hypersonic situations if we have slender configuration like wedge or cone. The force balance for a fluid particle in the presence of centrifugal force is shown in Fig. 10.1. However, if we have blunt nosed configuration then the pressure on the wall should be lower than that of predicted by Newtonian theory by the virtue of centrifugal force applied on the fluid particle as shown in Fig. 10.1.

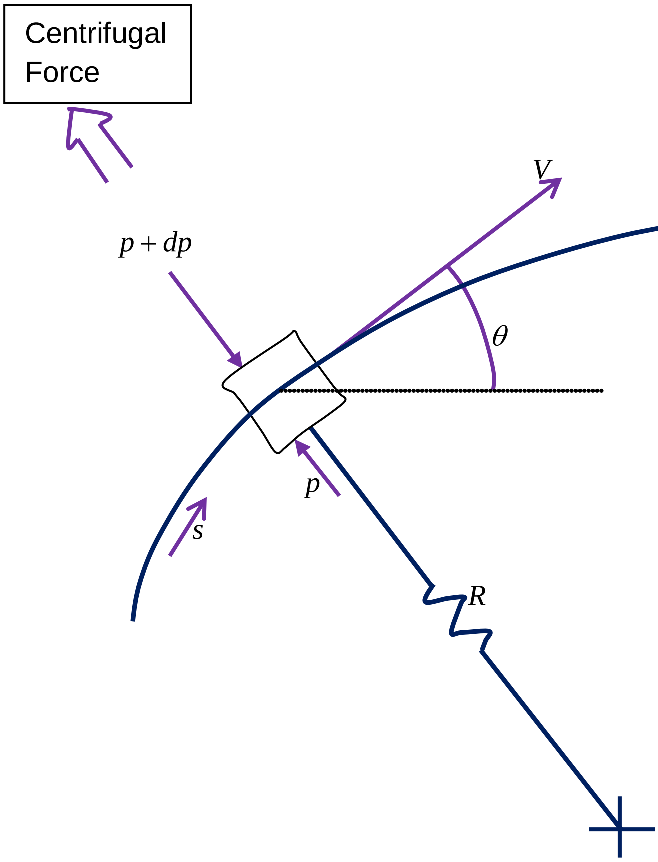

Fig. 10.1 Illustration of presence of centrifugal force for flow over blunt bodies [1].

Consider a hypersonic flow over a typical blunt body configuration as shown in Fig. 10.2.

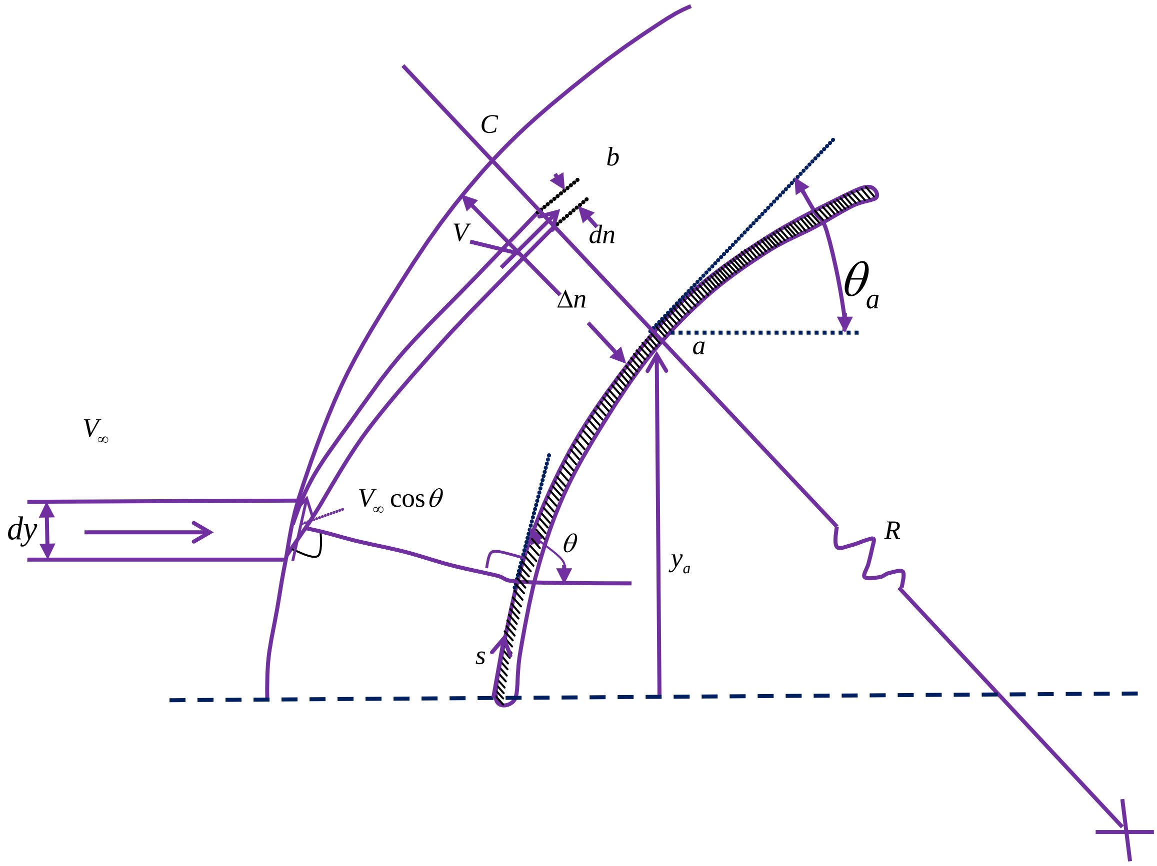

Fig. 10.2. Schematic of blunt body flow for Newtonian-Busemann theory [1]

Let's consider a stream tube of pre-shock height dy. This same stream tube gets deflected by angle θ behind the shock and earns thickness dn. Suppose 'R' is the radius of the curvature faced by the fluid particles passing this tube. This radius will be much more in comparison with the shock layer thickness. Therefore we can assume that all particles in shock layer face some curvature for calculation of the centrifugal force as,

$$\frac{\partial P}{\partial \eta} = \frac{\rho V^2}{R}$$

Here

Pressure (P)=f(S,η)

S = coordinate along stream tube

η = coordinate along normal to the stream tube

Hence,

$$dP = \frac{\rho V^2}{R} d\eta$$

We can integrate this expression along the thickness of the shock layer (line a-b-c) while considering the complete freestream instead of a stream tube. Therefore,

$$\int_a^c dP = \int_0^{\Delta\eta} \frac{\rho V^2}{R} d\eta$$

Here Δη is the shock layer thickness in the direction normal from point a on the blunt body. Hence,

$$p_c - p_a = \int_0^{\Delta\eta} \frac{\rho V^2}{R} d\eta$$

However we know from mass conservation for the stream tube that $\rho_\infty V_\infty dy = \rho V d\eta$

Therefore,

$$p_c - p_a = \int_o^{\Delta\eta} \rho_\infty V_\infty \frac{V}{R} dy$$

We have to change the limits of this integration since integration is with respect to y. Therefore we have to express Δη (shock layer thickness) in terms of y co-ordinate. Here we are considering a very thick stream tube in the freestream by increasing dy, such that it extends from the stagnation streamline till point c. Therefore the lower limit of integration will still be zero and upper limit will be y co-ordinate of point c.

$$p_c - p_a = \int_0^{y_c} \rho_\infty V_\infty \frac{V}{R} dy$$

Since $y_c = y_a + \Delta\eta \cos\theta_a$, so,

$$p_c - p_a = \int_0^{y_a + \Delta\eta\cos\theta_a} \rho_\infty V_\infty \frac{V}{R} dy$$

If we assume $y_a >> \Delta\eta \cos\theta_a$ then,

$$p_c - p_a = \int_0^{y_a} \rho_\infty V_\infty \frac{V}{R} dy$$

We know that tangential component of velocity is conserved across the shock hence this is the velocity which should be considered for balancing of centrifugal force with pressure. Therefore lets take $V = V_\infty \cos\theta$ and R as constant,

$$p_c - p_a = \int_0^{y_a} \rho_\infty V_\infty V_\infty \frac{\cos\theta}{R} dy$$

$$p_c - p_a = \left\{ \int_0^{y_a} \cos\theta . dy \right\} . \frac{\rho_\infty V_\infty{}^2}{R} \qquad (10.3)$$

Here we can express the radius of curvature (R) in terms of known parameters as,

$$R\frac{d\theta}{dS} = -1$$

$$R\left(\frac{d\theta}{dS}\right)_a = -1$$

$$dS = \frac{dy}{\sin(\theta)}$$

$$R = \frac{-1}{\left(\frac{d\theta}{dy}\right)_a \sin\theta_a}$$

Using this expression , we can re-write the Eq. 10.3 as,

$$p_c - p_a = -\rho_\infty V_\infty^{\ 2}\left(\frac{d\theta}{dy}\right)_a \sin\theta_a \int_0^{y_a} \cos\theta dy$$

$$p_a = p_c + \rho_\infty V_\infty^{\ 2}\left(\frac{d\theta}{dy}\right)_a \sin\theta_a \int_0^{y_a} \cos\theta dy$$

$$p_a - p_\infty = p_c - p_\infty + \rho_\infty V_\infty^{\ 2}\left(\frac{d\theta}{dy}\right)_a \sin\theta_a \int_0^{y_a} \cos\theta dy$$

$$\frac{p_a - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^{\ 2}} = \frac{p_c - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^{\ 2}} + \frac{\rho_\infty V_\infty^{\ 2}\left(\frac{d\theta}{dy}\right)_a \sin\theta_a \int_0^{y_a} \cos\theta dy}{\frac{1}{2}\rho_\infty V_\infty^{\ 2}}$$

$$c_{p_a} = c_{p_c} + 2\left(\frac{d\theta}{dy}\right)_a \sin\theta_a \int_0^{y_a} \cos\theta dy \qquad (10.4)$$

This is the final expression for pressure coefficient using Newtonian-Bwemann theory which is valid for 2D objects.

Some basic features of this theory are

1. It accounts for centrifugal force
2. It assumes small shock layer thickness as compared with body radius
3. It does not predict the pressure well, hence not advisable in general.
4. It is not truly local indication method since depends on upstream angles (in integration)

$$c_{p_c} = 2\sin^2\theta_a$$

Among all three Newton’s methods

1 Direct Newton’s method is more suitable for slender bodies like wedged or cones

2 Modified Newton’s method is suitable for blunt bodies.

3 Newtonian Busemann theory with centrifugal effect is not practically useful.

Lecture-11: Tangent wedge and tangent cone methods

11.1. Tangent wedge method

It is another local inclination method for 2D objects with attached shock condition. Hence this method if suitable for sharp nosed bodies with sufficiently high Mach numbers to maintain the shock attached to the nose. We can evaluate the pressure at any point on the object using this method. This calculation is strictly based on the local surface inclination or flow deflection angle. This theory is based on intuition rather on particular scientific derivation. Moreover the prediction made by this theory is encouraging at high Mach number.

Consider the 2D object shown in Fig. 11.1 for demonstration of process of this method.

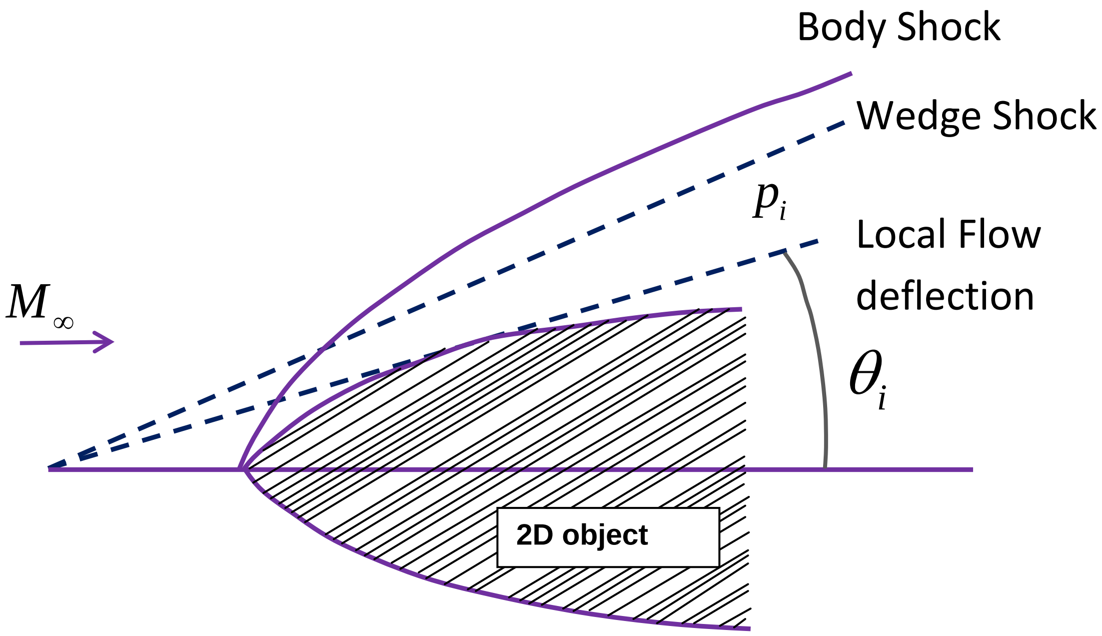

Fig. 11.1 Schematic for Tangent Wedge method [1]

Following procedure should be adopted for calculation of pressure at any point 'i'.

1. Draw tangent at the nose of the wedge and find out of the angle made by the tangent in reference with the freestream velocity vector.
2. Using this angle and freestream Mach number and then calculate the probable oblique shock angle.

3. If this shock angle is less than the maximum shock angle for that freestream Mach number then use Tangent Wedge method for calculation of pressure at any point on the 2D object.

3a. Select any point ‘i’ on the surface of the 2D object and draw tangent to the 2D surface at that point. Here we assume that the give 2D object is locally replaced by the equivalent wedge.

3b. From the known wedge angle and freestream Mach number use oblique shock relation to calculate the pressure at that point.

4. Follow this procedure at all points on the wedge surface and calculate the pressure on it.

11.3. Tangent cone method

This method is the obvious extension of tangent wedge method for 3D objects with attached shock condition. In this technique same procedure is to be followed however for 3D objects. Instead of the assumption of wedge, we have to assume presence of an equivalent cone at all the points on the surface of the 3D object. The simple difference exits due to the fact that there exists no analytical technique for evaluation of pressure using oblique shock relation for cone and we have to solve for the Taylor Maccoll equation to arrive at the shock angle for a given cone angle. Though this procedure is time consuming, it is simple to implement for estimation of wall pressure.

11.4. *Shock and expansion method*

It is also a local surface inclination method like tangent wedge, cone and the Newton's methods. It is applicable for only attached shock conditions like in tangent wedge and cone method. For implementation of this procedure consider 2D body with attached shock condition as shown in Fig 11.2.

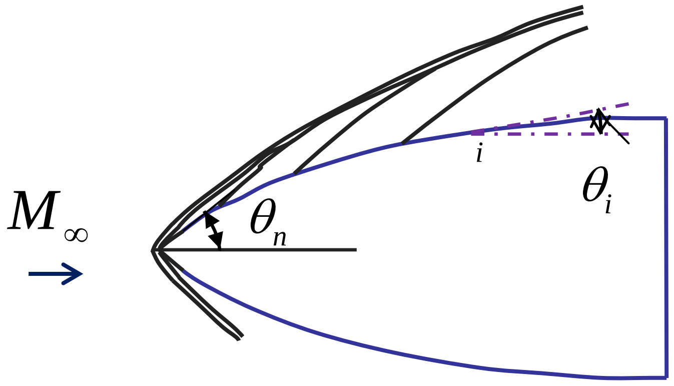

Fig. 11.2 Schematic for Shock and Expansion method [1].

Following procedure should be followed to implement the shock and expansion method

1. Calculate flow deflection angle at the nose for the (θ_n) 2D object.
2. Calculate Mach number and pressure at the nose using oblique shock relations for the known wedge angle (angle of flow deflection at nose) and freestream Mach number.
3. Here onwards use Prandtl Mayer relation for prediction of Mach number at any point on the surface.
4. Select a point just downstream of the nose and calculate flow deflection angle at that point using slope of the tangent at the same point.
5. Assume the flow to be expanding between nose to the selected point and calculate the Mach number at the chosen point using,

$$\theta_n - \theta_i = \Delta\theta = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_n^{\,2}-1\right)} - \tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_i^{\,2}-1\right)} - \left[\tan^{-1}\sqrt{M_n^{\,2}-1}\right] - \tan^{-1}\sqrt{M_i^{\,2}-1}$$

Here M_n is the Mach number at nose, θ_n is deflection angle at nose and θ_i & M_i are flow deflection & Mach number at any point just down stream of nose

6. Thus obtained Mi can be used to evaluate the rest of properties of the flow using isentropic relations.
7. Thus obtain properties at all the points on the surface. We can as well use the properties of point 'i' to calculate the properties at a point just downstream of it instead of nose properties.

This methodology can also, be used for axi-symmetric configurations also by using freestream Mach number and semi apex angle of cone. This theory does not consider any reflection of expansion fan from body shock to alter the pressure on body. Hence it creates approximations for low Mach number flows where interaction becomes inevitable. However results from this theory for hypersonic flow are encouraging since wave angle decreases with increase in Mach number for same deflection angle and interaction gets avoided as shown in Fig. 11.3.

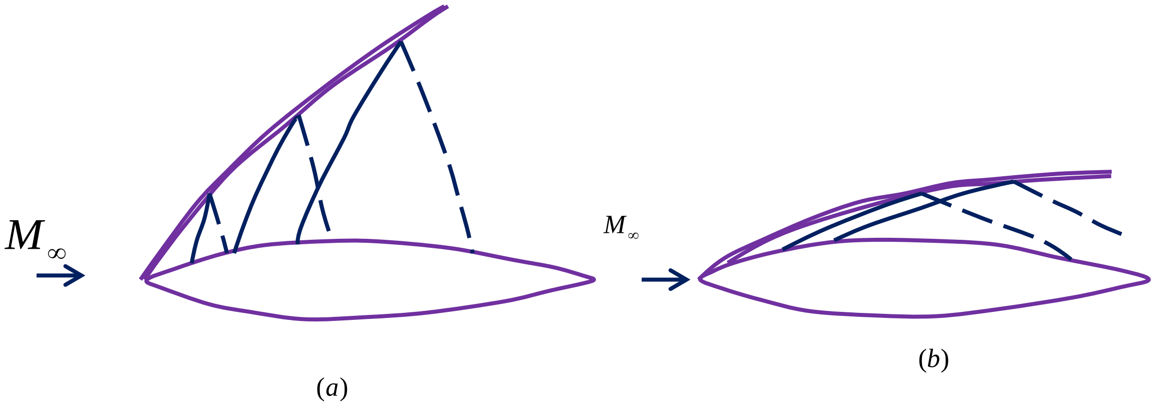

Fig. 11.3. Illustration for interaction of waves for (a) supersonic flows and (b) hypersonic flows [1].