Module-4: Approximate and exact methods for hypersonic inviscid flows

Lecture-12: Hypersonic Mach number independence principle

12.1 Introduction

For major considerations in hypersonic flow, detailed analysis of the flow-field is required. The details of any flow-field are governed by a system of conservation equations which can be expressed in either integral or partial differential equation form. There are approximate solutions of these equations for various hypersonic applications. These approximate solutions are based upon pure theories of fluid dynamics. Solution for the hypersonic flowfield was available before the advent of high speed computers through the theoretical considerations, analytical formulations or approximate techniques. However, many of these older analyses, all of which involved some approximations to allow the solution of the governing equations, are just as relevant to the modern hypersonics of today as they were in 1950s. One major advantage of approximate theories of hypersonic flows is that they illustrate the effect of various parameters on the physical results more than the exact (numerical) solution methods.

12.2 The Governing Equations

As we had seen, the governing equations for hypersonic flow are,

Continuity : $$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \qquad (12.1)$$

X Momentum : $$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} \qquad (12.2)$$

Y Momentum : $$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} \qquad (12.3)$$

Z Momentum : $$\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z} \qquad (12.4)$$

Energy : $$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}=0 \qquad (12.5)$$

The above equations are governing differential equations for inviscid compressible flows. They are named as Euler Equations. The entropy from the energy equation can again be replaced by $\frac{p}{\rho^{\gamma}}$=Constant for isentropic process in an ideal gas along a streamline. Hence, the energy equation can be written as,

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)+u\frac{\partial}{\partial x}\left(\frac{p}{\rho^{\gamma}}\right)+v\frac{\partial}{\partial y}\left(\frac{p}{\rho^{\gamma}}\right)+w\frac{\partial}{\partial z}\left(\frac{p}{\rho^{\gamma}}\right)=0 \qquad (12.6)$$

The solution of the above equations for a problem depends on the boundary and the initial conditions for that problem.

12.3 Mach Number Independence

At high Mach Numbers, certain aerodynamic quantities, such as pressure coefficient, lift and wave drag coefficients and flow-field structure (such as shock wave shapes and Mach wave patterns) become essentially independent of Mach number. This is called Mach Number Independence Principle. To understand it explicitly, let us depict the non-dimensionalization of the above mentioned governing equations and corresponding boundary conditions.

$$\overline{x}=\frac{x}{l},\ \overline{y}=\frac{y}{l},\ \overline{z}=\frac{z}{l},$$

$$\overline{u}=\frac{u}{V_\infty},\ \overline{v}=\frac{v}{V_\infty},\ \overline{w}=\frac{w}{V_\infty},$$

$$\overline{p}=\frac{p}{\rho_\infty V_\infty^2},\ \overline{\rho}=\frac{\rho}{\rho_\infty},$$

Where, '*l*' denotes a characteristic length of the flow, ρ_∞ and V_∞ are freestream density and free stream velocity respectively. Use of this non-dimensionalization for steady state continuity equation is,

$$\frac{\rho_\infty V_\infty}{l}\left[\frac{\partial\left(\frac{\rho}{\rho_\infty}\cdot\frac{u}{V_\infty}\right)}{\partial\left(\frac{x}{l}\right)}+\frac{\partial\left(\frac{\rho}{\rho_\infty}\cdot\frac{v}{V_\infty}\right)}{\partial\left(\frac{y}{l}\right)}+\frac{\partial\left(\frac{\rho}{\rho_\infty}\cdot\frac{w}{V_\infty}\right)}{\partial\left(\frac{z}{l}\right)}\right]=0$$

$$\frac{\partial}{\partial\left(\bar{x}\right)}\left(\bar{\rho}.\bar{u}\right)+\frac{\partial}{\partial\left(\bar{y}\right)}\left(\bar{\rho}.\bar{v}\right)+\frac{\partial}{\partial\left(\bar{z}\right)}\left(\bar{\rho}.\bar{w}\right)=0 \qquad (12.7)$$

Similarly we can write for other governing equations,

$$\bar{\rho}.\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{\rho}.\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}+\bar{\rho}.\bar{w}\frac{\partial \bar{u}}{\partial z}=-\frac{\partial \bar{p}}{\partial \bar{x}} \qquad (12.8)$$

$$\bar{\rho}.\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{\rho}.\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}+\bar{\rho}.\bar{w}\frac{\partial \bar{v}}{\partial \bar{z}}=-\frac{\partial \bar{p}}{\partial \bar{y}} \qquad (12.9)$$

$$\bar{\rho}.\bar{u}\frac{\partial \bar{w}}{\partial \bar{x}}+\bar{\rho}.\bar{v}\frac{\partial \bar{w}}{\partial \bar{y}}+\bar{\rho}.\bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}=-\frac{\partial \bar{p}}{\partial z} \qquad (12.10)$$

$$\bar{u}\frac{\partial}{\partial \bar{x}}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right)+\bar{v}\frac{\partial}{\partial \bar{y}}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right)+\bar{w}\frac{\partial}{\partial \bar{z}}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right)=0 \qquad (12.11)$$

Any particular solution of these non-dimensional governing equations is again governed by the boundary conditions for flow over a hypersonic body. We know that the boundary condition for the steady inviscid hypersonic flow on the body surface is, $\vec{V}.\hat{n}=0$, which necessarily means that the normal component of velocity on the wall is zero. Therefore,

$$un_x + vn_y + wn_z = 0$$

The u,v and w are x,y and z directional velocities and n_x, n_y, n_z are the direction cosines The non-dimensional form of this boundary condition is,

$$\bar{u}\,n_x + \bar{v}\,n_y + \bar{w}\,n_z = 0 \qquad (12.13)$$

The flow-field for the problem is bounded on one side by the body surface and on the other side by the bow shock wave. The boundary conditions behind the shock are given by the oblique shock properties.

Lecture-13: Hypersonic Mach number independence principle

13.1 Mach Number Independence (continued)

We intend to solve for the hypersonic flowfield using the known governing equations and the boundary conditions at the wall along with the boundary conditions for the shock. Here we assume that the location and shape of the shock is known to us and we are intended to solve for the flowfield in the shock layer or the volume between shock and body. This technique is called as shock fitting technique. We can obtain correlations or use prior flow visualization data for shock shape. Therefore such technique can be used for analyzing the flowfield in the shock layer.

Oblique shock relations can be used for boundary conditions at shock location.

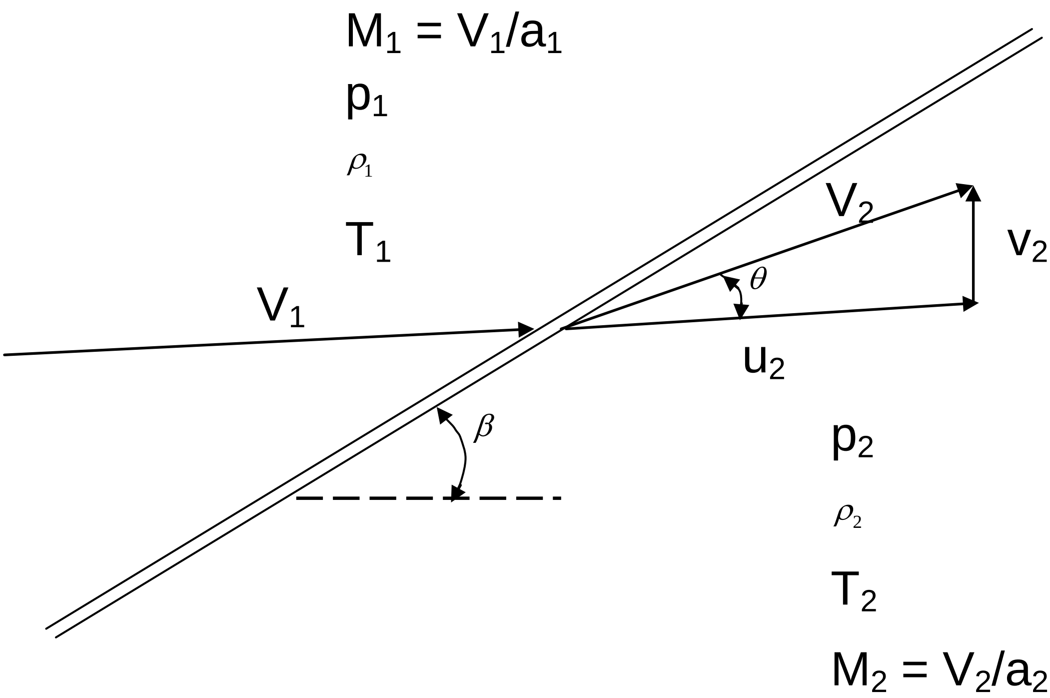

Fig. 12.1 Schematic of the oblique shock.

A typical oblique shock and corresponding properties are shown in Fig. 12.1. Using known freestream properties we can calculate the properties behind the shock. These post shock properties can be used as the boundary conditions at the shock to solve the governing equations. Following oblique shock relations are to be used for this purpose.

$$\frac{p_2}{p_\infty} = 1 + \frac{2\gamma}{\gamma+1}\left(M_\infty^2 Sin^2\beta - 1\right) \qquad (13.1)$$

$$\frac{\rho_2}{\rho_\alpha} = \frac{(\gamma+1)M_\infty^2 Sin^2\beta}{(\gamma-1)M_\infty^2 Sin^2\beta + 2} \qquad (13.2)$$

$$\frac{u_2}{V_\infty} = 1 - \frac{2(M_\infty^2 Sin^2\beta - 1)}{(\gamma+1)M_\infty^2} \qquad (13.3)$$

$$\frac{v_2}{V_\infty} = \frac{2(M_\infty^2 Sin^2\beta - 1)Cot\beta}{(\gamma+1)M_\infty^2} \qquad (13.4)$$

However the governing equations are in non-dimensional, hence we have to non-dimensionalze the post shock variables with the same reference variables.

Now, $\frac{p_2}{p_\infty} = \frac{\overline{p}_2 \rho_\infty V_\infty^2}{p_\infty}$

$$\frac{p_2}{p_\infty} = \frac{\overline{p}_2 . V_\infty^2}{RT_\infty}$$

$$\frac{p_2}{p_\infty} = \frac{\gamma \overline{p}_2 . V_\infty^2}{\gamma RT_\infty}$$

$$\frac{p_2}{p_\infty} = \frac{\gamma \overline{p}_2 . V_\infty^2}{a^2{}_\infty}$$

$$\frac{p_2}{p_\infty} = \gamma \overline{p}_2 . M_\infty^2$$

Therefore the Eq. 13.1 can be re-written as,

$$\bar{p}_2 = \frac{1}{\gamma M_\infty^2} + \frac{2\gamma}{\gamma+1}\left(\frac{M_\infty^2 Sin^2\beta - 1}{\gamma M_\infty^2}\right)$$

Or $$\bar{p}_2 = \frac{1}{\gamma M_\infty^2} + \frac{2}{\gamma+1}\left(Sin^2\beta - \frac{1}{M_\infty^2}\right) \qquad (13.5)$$

Also, $$\bar{\rho}_2 = \frac{(\gamma+1) M_\infty^2 Sin^2\beta}{(\gamma-1) M_\infty^2 Sin^2\beta + 2} \qquad (13.6)$$

$$\bar{u}_2 = 1 - \frac{2(M_\infty^2 Sin^2\beta - 1)}{(\gamma+1) M_\infty^2} \qquad (13.7)$$

$$\bar{v}_2 = \frac{2(M_\infty^2 Sin^2\beta - 1) Cot\beta}{(\gamma+1) M_\infty^2} \qquad (13.8)$$

If we take the limit of Mach number tends to ∞. Therefore in the limit as $M_\infty \to \infty$.

$$\bar{p}_2 \to \frac{2Sin^2\beta}{\gamma+1} \qquad (13.9)$$

$$\bar{\rho}_2 \to \frac{\gamma+1}{\gamma-1} \qquad (13.10)$$

$$\bar{u}_2 \to 1 - \frac{2Sin^2\beta}{\gamma+1} \qquad (13.11)$$

$$\bar{v}_2 \to \frac{2Sin^2\beta Cot\beta}{\gamma+1} = \frac{Sin2\beta}{\gamma+1} \qquad (13.12)$$

Here we can observe the fact that the governing equations in their non-dimensional form (Eq. 12.7-12.11) are independent of Mach number. The boundary condition at the wall (Eq. 12.13) and the boundary conditions at the shock (Eq. 13.9-13.12) are also independent of Mach number. Therefore the expected flowfield in the shock layer should also be independent of Mach number. This fact is called as Mach number independence principle. It is valid for very high Mach number inviscid flows. The above derived fact is about two dimensional flows. This independence appears at relatively lower Mach number for blunt bodies in comparison with slender ones due to the fact that the shock angle is very high and all the terms in the Eq. 13.3 to 13.8 include $M_{\infty} Sin\beta$.

Lecture-14: The Hypersonic Small Disturbance Theory

14.1 The Hypersonic Small Disturbance Theory

Some special applications in the hypersonic speed regime need aircraft configurations of low drag and high lift or high lift to drag (L/D) ratio. Therefore, in such hypersonic applications, slender body configurations are generally preferred. Governing equations can be specially derived for this constraint. The formulation in this case is referred to as Hypersonic Small Disturbance Equations.

Consider a typical slender body as shown in Fig. 14.1

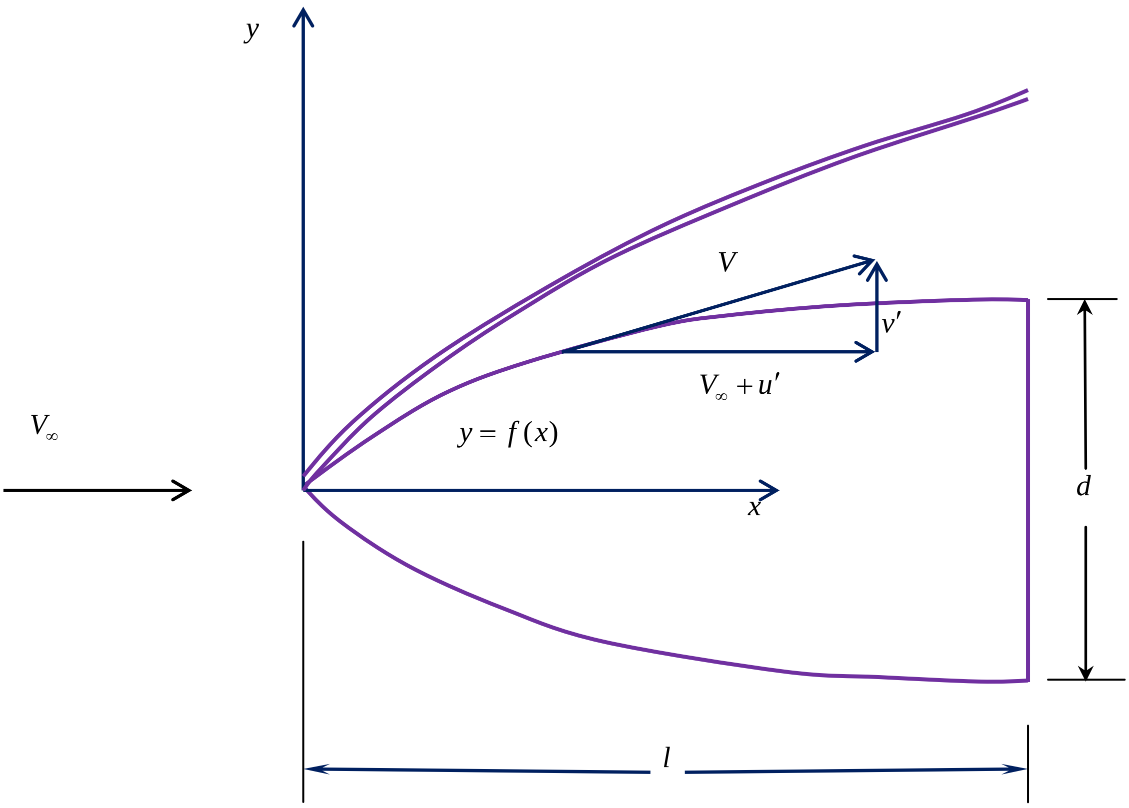

Fig 14.1: Typical slender body considered for high L/D [1].

Here the freestream velocity V_∞ gets changed behind the shock and then on the body. Let us assume that the change in this velocity is more while cross the shock in comparison with the change taking place during the travel of the fluid over the body. Let u' and v' be the change in x and y components of velocity vector. Thus the velocity at any point on the surface of the body can be given by,

$$\overline{V} = \vec{u} + \vec{v} = \hat{i}u + \hat{j}v = \hat{i}u + \hat{j}v = \hat{i}(V_\infty + u') + \hat{j}v'$$

Here, freestream velocity V_∞ is assumed to be aligned with X-axis. The changes in the velocity, i.e. u' and v' are called perturbation velocities. These changes might not be always small. However for hypersonic Flow over slender body we are assuming these changes are small in comparison with the freestream velocity.

$$u' << V_\infty$$

and $v' << V_\infty$

But u' and v' may not be < a_∞. Let the surface of the slender body follow a profile mathematically represented as $y = f(x)$. Since the velocity $\vec{V}$, at any point on the body surface, is tangential to the surface at that point, slope of the velocity and slope of the tangent to the surface are same. Hence this tangency condition can be written as,

$$\frac{v'}{V_\infty + u'} = \left(\frac{dy}{dx}\right)_{\text{on the body}} \qquad (14.1)$$

Since the configuration under consideration is slender, the slope at any point is of the order of ratio of length to diameter of the body.

$$\frac{dy}{dx} = O\left(\frac{l}{d}\right) \qquad (14.2)$$

Hence, $\frac{dy}{dx} = \tau$ =Slenderness Ratio

Therefore, $\frac{v'}{V_\infty + u'} = \frac{dy}{dx} = O(\tau)$ (14.3)

However, $u' << V_\infty$, which leads to,

$$\frac{v'}{V_\infty} = O(\tau) \qquad (14.4)$$

Let a_∞ be the freestream speed of sound,

$$\frac{v'}{a_\infty} = \frac{V_\infty}{a_\infty} O(\tau)$$

$$\frac{v'}{a_\infty} = O(M_\infty \tau) \qquad (14.5)$$

Hence, Strength of the disturbance in the flow relative to a_∞ ($\frac{v'}{a_\infty}$) is of the order of $M_\infty \tau$. This expression also provides information that $M_\infty \tau$ is a hypersonic similarity parameter. This information is fetched only from the boundary conditions.

We can express the steady Euler Equations in terms of the perturbation velocities, *u'* and *v'* for steady flow,

$$\frac{\partial[\rho(V_\infty + u')]}{\partial x} + \frac{\partial(\rho v')}{\partial y} + \frac{\partial(\rho w')}{\partial z} = 0 \qquad (14.6)$$

$$\rho(V_\infty + u')\frac{\partial(V_\infty + u')}{\partial x} + \rho v'\frac{\partial(V_\infty + u')}{\partial y} + \rho w'\frac{\partial(V_\infty + u')}{\partial z} = -\frac{\partial p}{\partial x} \qquad (14.7)$$

$$\rho(V_\infty + u')\frac{\partial(v')}{\partial x} + \rho v'\frac{\partial(v')}{\partial y} + \rho w'\frac{\partial(v')}{\partial z} = -\frac{\partial p}{\partial y} \qquad (14.8)$$

$$\rho(V_\infty + u')\frac{\partial(w')}{\partial x} + \rho v'\frac{\partial(w')}{\partial y} + \rho w'\frac{\partial(w')}{\partial z} = -\frac{\partial p}{\partial z} \qquad (14.9)$$

$$(V_\infty + u')\frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right) + v'\frac{\partial}{\partial y}\left(\frac{p}{\rho^\gamma}\right) + w'\frac{\partial}{\partial z}\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad (14.10)$$

Lecture-15: The Hypersonic Small Disturbance Theory

15.1 The Hypersonic Small Disturbance Theory (continued)

The governing equations especially derived for the small disturbance theory (Eq. 14.6 to 14.10) can be used to formulate the same. However boundary conditions at shock and at the wall are to be evaluated in the same format to solve these equations. In view of this, Let us derive the expression for boundary conditions (Eq. 13.9 to 13.12). For slender body at hypersonic speeds, both shock wave angle 'β' and the deflection angle 'θ' can be assumed to be small and hence equal. Therefore,

$$Sin\beta \simeq Sin\theta \simeq \theta \simeq \frac{dy}{dx} \simeq \tau$$

We know that,

$$\frac{p_2}{p_\infty} = 1 + \frac{2\gamma}{\gamma+1}\left(M_\infty^2 Sin^2\beta - 1\right)$$

For very high Mach numbers,

$$\frac{p_2}{p_\infty} \to \frac{2\gamma}{\gamma+1} M_\infty^2 Sin^2\beta$$

$$\frac{p_2}{p_\infty} \to O[M_\infty^2 \tau^2] \qquad (15.1)$$

$$p_2 \to O[M_\infty^2 \tau^2 p_\infty] \qquad (15.2)$$

Hence the non-dimensional pressure of unity order of magnitude is,

$$\bar{p} = \frac{p}{\gamma M_\infty^2 \tau^2 p_\infty}$$

On the same lines for very high Mach numbers,

$$\frac{\rho_2}{\rho_\alpha} \to \frac{(\gamma+1)}{(\gamma-1)}$$

The order of magnitude of this ratio is also unity. Since, for $\gamma = 1.4$, $\frac{\rho_2}{\rho_\alpha} \to 6$, which is $O(1)$.

$$\frac{u_2}{V_\infty} \to 1 - \frac{2Sin^2\beta}{(\gamma+1)}$$

Let us define change in *x*-component of the velocity across the oblique shock as,

$$\Delta u = V_\infty - u_2$$

So, $\frac{\Delta u}{V_\infty} = \frac{V_\infty - u_2}{V_\infty} \to \frac{2Sin^2\beta}{\gamma+1} \to O(\tau^2)$

This implies that the non-dimensional perturbation velocity $\overline{u'}$ (change in velocity in the x-direction) should be $\overline{u'} = \frac{u'}{(V_\infty \tau^2)}$ in order to have unity order of magnitude.

$$\frac{v_2}{V_\infty} \to \frac{Sin2\beta}{(\gamma+1)} \to O(\tau)$$

So, the non-dimensional perturbation velocity $\overline{v'}$ will be $\overline{v'} = \frac{v'}{V_\infty \tau}$ to make it on the order of magnitude of unity.

Therefore it is very much evident that, $\Delta v \to O(\tau)V_\infty$ and $\Delta u = O(\tau^2)V_\infty$. This fact leads to $\Delta v > \Delta u$ since $\tau << 1$.

Therefore the new non-dimensional form of flow variables is,

$$\overline{x} = \frac{x}{l},\ \overline{y} = \frac{y}{l\tau},\ \overline{z} = \frac{z}{l\tau},$$

$$\overline{u'} = \frac{u'}{V_\infty \tau^2},\ \overline{v'} = \frac{v'}{V_\infty \tau},\ \overline{w'} = \frac{w'}{V_\infty \tau},$$

$$\overline{p} = \frac{p}{\gamma M_\infty^2 \tau^2 p_\infty},\ \overline{\rho} = \frac{\rho}{\rho_\infty}$$

In view of this non-dimensionalization, we have to reformulate the governing equations. The mass conservation equation is,

$$\frac{\partial[\rho(V_\infty + u')]}{\partial x} + \frac{\partial(\rho v')}{\partial y} + \frac{\partial(\rho w')}{\partial z} = 0$$

$$\frac{\partial[\bar{\rho}(\frac{1}{\tau^2} + \bar{u}')]}{\partial \bar{x}}\left(\frac{\rho_\infty V_\infty \tau^2}{l}\right) + \frac{\partial(\bar{\rho}\,\bar{v}')}{\partial \bar{y}}\left(\frac{\rho_\infty V_\infty \tau}{l\tau}\right) + \frac{\partial(\bar{\rho}\,\bar{w}')}{\partial \bar{z}}\left(\frac{\rho_\infty V_\infty \tau}{l\tau}\right) = 0$$

$$\frac{\partial[\bar{\rho}(1 + \bar{u}'\tau^2)]}{\partial \bar{x}} + \frac{\partial(\bar{\rho}\,\bar{v}')}{\partial \bar{y}} + \frac{\partial(\bar{\rho}\,\bar{w}')}{\partial \bar{z}} = 0 \qquad (15.3)$$

Proceeding in similar way we get the x-momentum equation as,

$$\frac{\partial[\bar{\rho}\,\rho_\infty(1 + \frac{u'}{V_\infty})]V_\infty}{\partial(\bar{x}\,l)} + \frac{\partial[\bar{\rho}\,\rho_\infty\,\bar{v}'\,V_\infty\tau]}{\partial(\bar{y}\,l\tau)} + \frac{\partial[\bar{\rho}\,\rho_\infty\,\bar{w}'V_\infty\tau]}{\partial(\bar{z}\,l\tau)} = 0$$

$$\bar{\rho}(1 + \bar{u}'\tau^2)\frac{\partial \bar{u}'}{\partial \bar{x}} + \bar{\rho}\,\bar{v}'\frac{\partial \bar{u}'}{\partial \bar{y}} + \bar{\rho}\,\bar{w}'\frac{\partial \bar{u}'}{\partial \bar{z}} = -\frac{\partial \bar{p}}{\partial \bar{x}} \qquad (15.4)$$

The y-momentum equation is,

$$\bar{\rho}(1 + \bar{u}'\tau^2)\frac{\partial \bar{v}'}{\partial \bar{x}} + \bar{\rho}\,\bar{v}'\frac{\partial \bar{v}'}{\partial \bar{y}} + \bar{\rho}\,\bar{w}'\frac{\partial \bar{v}'}{\partial \bar{z}} = -\frac{\partial \bar{p}}{\partial \bar{y}} \qquad (15.5)$$

The z-momentum equation is,

$$\bar{\rho}(1 + \bar{u}'\tau^2)\frac{\partial \bar{w}'}{\partial \bar{x}} + \bar{\rho}\,\bar{v}'\frac{\partial \bar{w}'}{\partial \bar{y}} + \bar{\rho}\,\bar{w}'\frac{\partial \bar{w}'}{\partial \bar{z}} = -\frac{\partial \bar{p}}{\partial \bar{z}} \qquad (15.6)$$

And the energy equation is,

$$(1 + \tau^2\,\bar{u}')\frac{\partial}{\partial x}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) + \bar{v}'\frac{\partial}{\partial y}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) + \bar{w}'\frac{\partial}{\partial z}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) = 0 \qquad (15.7)$$

Here all the terms involved in making the governing equations (15.3 to 15.7) are of unity order, $O(1)$. We can simplify these equation using the known fact that $\tau^2 << 1$. Therefore the simplified form of the same equations are,

$$\frac{\partial \bar{\rho}}{\partial \bar{x}} + \frac{\partial (\bar{\rho}\,\overline{v'})}{\partial \bar{y}} + \frac{\partial (\bar{\rho}\,\overline{w'})}{\partial \bar{z}} = 0 \tag{15.8}$$

$$\bar{\rho}\frac{\partial \overline{u'}}{\partial \bar{x}} + \bar{\rho}\,\overline{v'}\frac{\partial \overline{u'}}{\partial \bar{y}} + \bar{\rho}\,\overline{w'}\frac{\partial \overline{u'}}{\partial \bar{z}} = -\frac{\partial \bar{p}}{\partial \bar{x}} \tag{15.9}$$

$$\bar{\rho}\frac{\partial \overline{v'}}{\partial \bar{x}} + \bar{\rho}\,\overline{v'}\frac{\partial \overline{v'}}{\partial \bar{y}} + \bar{\rho}\,\overline{w'}\frac{\partial \overline{v'}}{\partial \bar{z}} = -\frac{\partial \bar{p}}{\partial \bar{y}} \tag{15.10}$$

$$\bar{\rho}\frac{\partial \overline{w'}}{\partial \bar{x}} + \bar{\rho}\,\overline{v'}\frac{\partial \overline{w'}}{\partial \bar{y}} + \bar{\rho}\,\overline{w'}\frac{\partial \overline{w'}}{\partial \bar{z}} = -\frac{\partial \bar{p}}{\partial \bar{z}} \tag{15.11}$$

$$\frac{\partial}{\partial \bar{x}}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) + \overline{v'}\frac{\partial}{\partial \bar{y}}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) + \overline{w'}\frac{\partial}{\partial \bar{z}}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) = 0 \tag{15.12}$$

These equations and the corresponding boundary equations of unity order comprise the equations for hypersonic small disturbance theory. Following inferences can be drawn from the this theory,

- These equations are limited to hypersonic flow over slender bodies.
- The parameter $\overline{u'}$ is decoupled from the system of equations. Once $\overline{v'}$, $\overline{w'}$, $\bar{\rho}$ and $\bar{p}$ are solved, $\overline{u'}$ can be found out.
- This reasserts the fact that, change in velocity in the flow direction is much smaller than the change in velocity perpendicular to the flow direction.

Lecture-16: The Hypersonic Equivalence Principle

16.1 The Hypersonic Equivalence Principle

Consider the Euler Equations for two-dimensional unsteady flow in y-z plane. Here x-axis is the direction of freestream velocity. Therefore the governing equations in y-z plane perpendicular to the freestream velocity can be written as,

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{16.1}$$

$$\rho\frac{\partial v}{\partial t}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y} \tag{16.2}$$

$$\rho\frac{\partial w}{\partial t}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z} \tag{16.3}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)+v\frac{\partial}{\partial y}\left(\frac{p}{\rho^{\gamma}}\right)+w\frac{\partial}{\partial z}\left(\frac{p}{\rho^{\gamma}}\right)=0 \tag{16.4}$$

Let us non-dimensionalize using following reference variables,

$$\overline{t}=\frac{t}{(l/V_{\infty})},\ \overline{y}=\frac{y}{l},\ \overline{z}=\frac{z}{l},$$

$$\overline{v}=\frac{v}{V_{\infty}},\ \overline{w}=\frac{w}{V_{\infty}},$$

$$\overline{p}=\frac{p}{\rho_{\infty}V_{\infty}^{2}},\ \overline{\rho}=\frac{\rho}{\rho_{\infty}}$$

Where, V_{∞} and ρ_{∞} can be treated as reference quantities.

Therefore the non-dimensional form of governing equations are,

$$\frac{\partial\overline{\rho}}{\partial\overline{t}}+\frac{\partial(\overline{\rho}\,\overline{v})}{\partial\overline{y}}+\frac{\partial(\overline{\rho}\,\overline{w})}{\partial\overline{z}}=0 \tag{16.5}$$

$$\overline{\rho}\frac{\partial \overline{v}}{\partial \overline{t}}+\overline{\rho}\,\overline{v}\frac{\partial \overline{v}}{\partial \overline{y}}+\overline{\rho}\,\overline{w}\frac{\partial \overline{w}}{\partial \overline{z}}=-\frac{\partial \overline{p}}{\partial \overline{y}} \tag{16.6}$$

$$\overline{\rho}\frac{\partial \overline{w}}{\partial \overline{t}}+\overline{\rho}\,\overline{v}\frac{\partial \overline{w}}{\partial \overline{y}}+\overline{\rho}\,\overline{w}\frac{\partial \overline{w}}{\partial \overline{z}}=-\frac{\partial \overline{p}}{\partial \overline{z}} \tag{16.7}$$

$$\frac{\partial}{\partial \overline{t}}\left(\frac{\overline{p}}{\overline{\rho}^{\gamma}}\right)+\overline{v}'\frac{\partial}{\partial \overline{y}}\left(\frac{\overline{p}}{\overline{\rho}^{\gamma}}\right)+\overline{w}'\frac{\partial}{\partial \overline{z}}\left(\frac{\overline{p}}{\overline{\rho}^{\gamma}}\right)=0 \tag{16.8}$$

Comparing these equations (16.5) to (16.8), with the earlier derived equations (15.8) to (15.12), it can be seen that these equations are identical if we ignore the variables involved in making them. However the equations (15.8) to (15.12) are for three dimensional steady flow and equations (16.5) to (16.8) are for two-dimensional unsteady flow. This information is helpful in depicting the hypersonic equivalence principle which states that, "The steady hypersonic flow over a slender object is equivalent to an unsteady flow in one less space dimension". This analogy can be visualized using the Fig. 11.1. Consider a slender body translating x-direction direction as shown in Fig. 11.1. The frozen moment at initial time is shown in two views, one in x-z plane and other in y-z plane, are placed side by side in the same figure. The view in x-z plane is the configuration of the slender body while view in the y-z plane is just a point at origin in the initial time. If V_∞ is the velocity of the object (in the negative x direction) then at any time later (say t_1), tip of the object moves in negative x direction by distance V_∞ x t_1. In this process the view of the object in x-z plane just gets translated as shown in Fig.11.1. However, the view of the same object in y-z plane is the section of the same object taken by y-z plane at origin or section at distance V_∞ x t_1 from the tip. Similarly after time t_2 the view the view in x-z plane gets further translated while view in y-z plane portrays the section of the object at the distance V_∞ x t_2 from the tip.

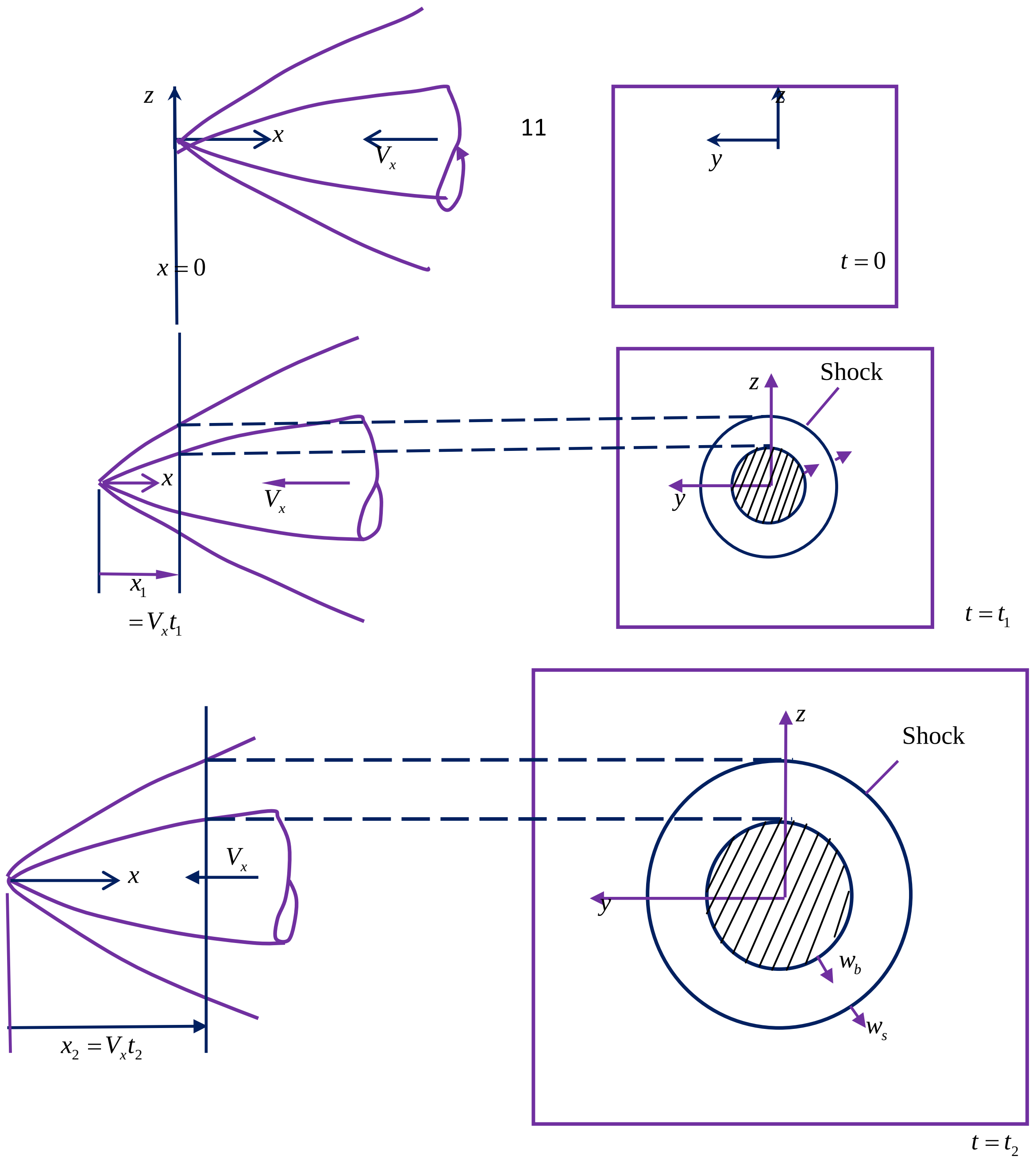

Fig 11.1: Illustration of the hypersonic Equivalence Principle [1]

This illustration makes it very much clear that study of three dimensional steady hypersonic flow over a slender object is equivalent to study two dimensional unsteady flow over the same. However we will have to view it differently. This equivalence principle forms the basis of the Blast Wave Theory.

Lecture-17: Blast Wave Theory

17.1. Blast Wave Theory

We have seen that the depiction of hypersonic equivalence principle is all about the equivalence between steady hypersonic flow over slender body and unsteady hypersonic flow in one lesser dimension. This unsteady shock motion can be realized by the instantaneous release of energy or a blast at the origin. An illustration in this regard is shown in Fig. 17.1 and 17.2 for blunt nosed cylinder and blunt nosed slab respectively.

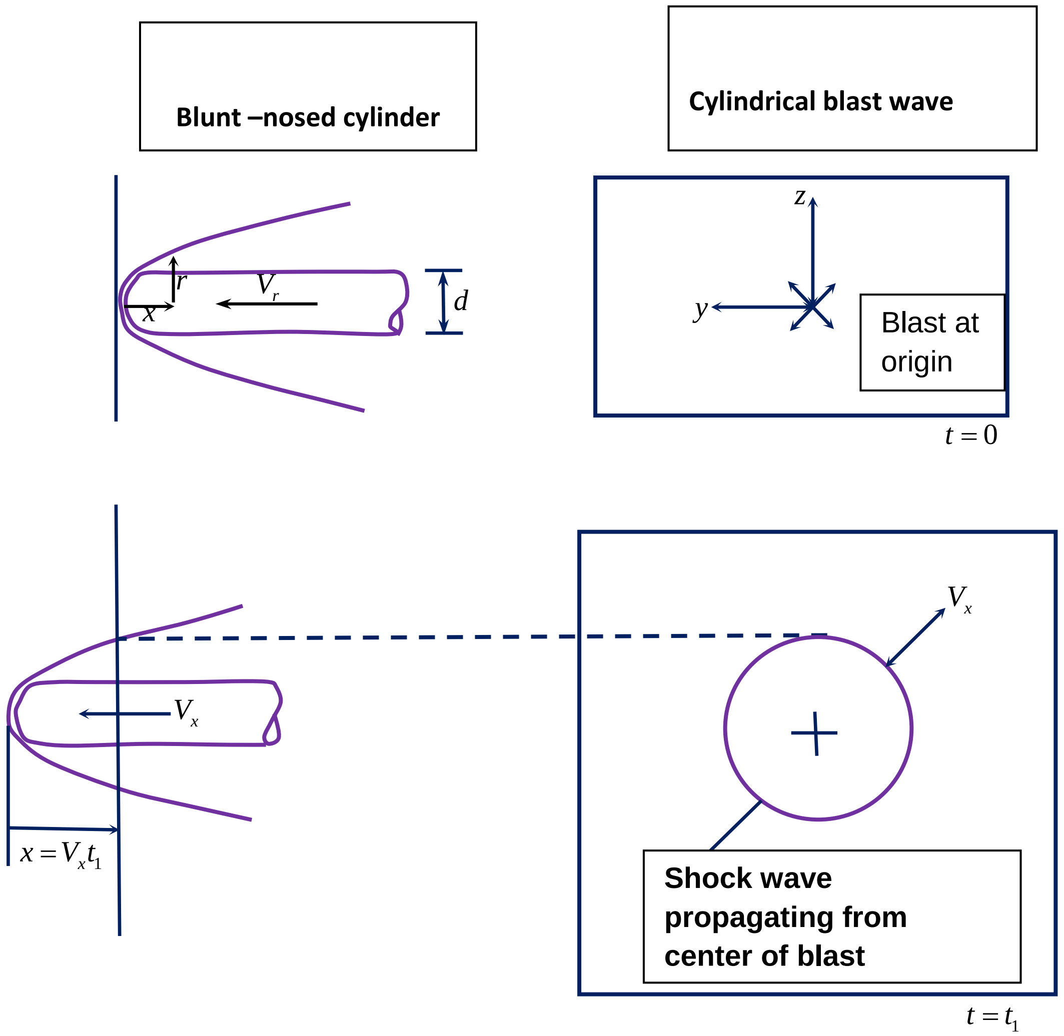

Fig.17.1: Blast wave analogy for a blunt-nosed cylinder [1]

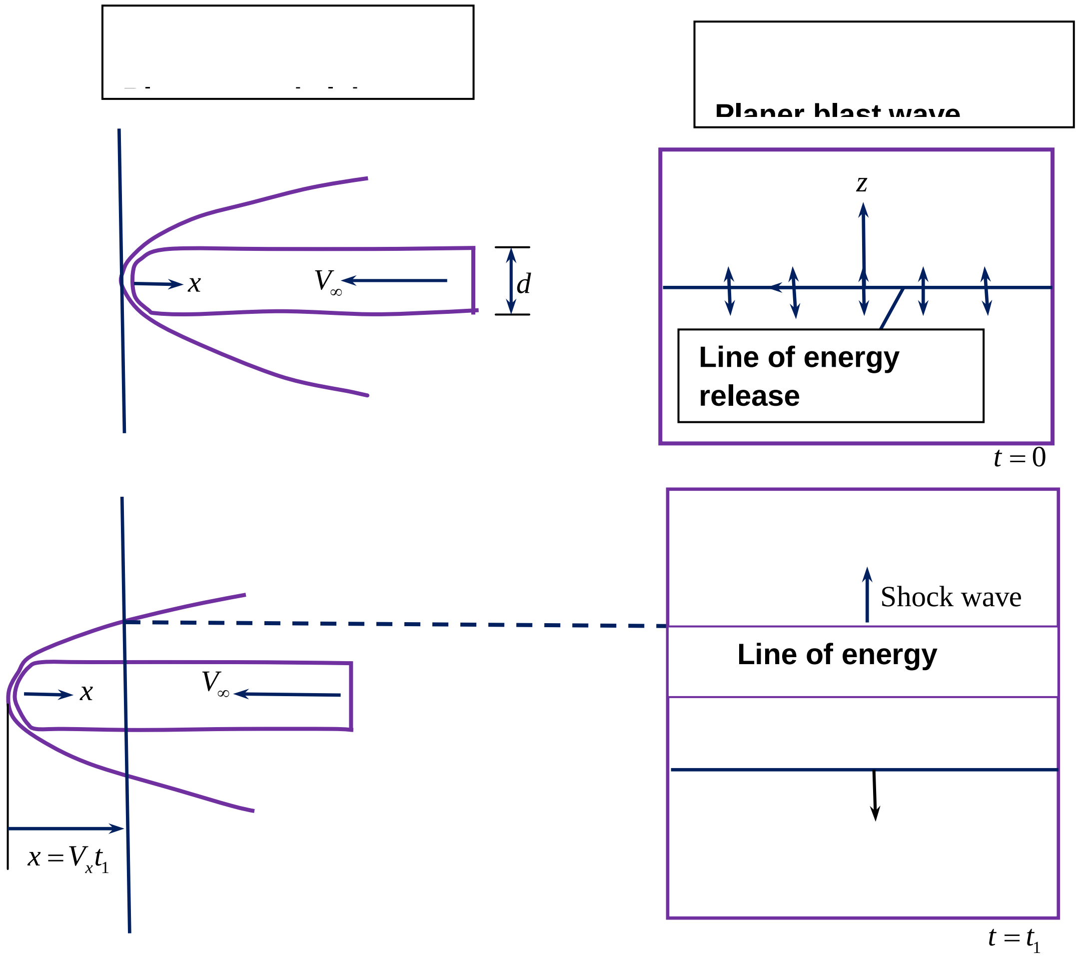

Fig. 17.2: Blast wave analogy for a blunt-nosed slab [1]

These two figures clearly suggest that the propagation of shock wave due to instantaneous energy release at the origin is equivalent with steady flow with the corresponding object. This propagation of shock wave due to energy release is called as blast wave. We can use this concept to estimate the pressure distribution, for examples on axisymmetric blunt nosed cylinders at hypersonic speeds, with the axis aligned in the direction of the flow, using the concept of blast wave. Direct analytical solutions are available for the planar and cylindrical blast waves for pressure near the center of explosion. Using such relations and analogy of blast wave we can evaluate the pressure distribution on the body. This theory is called as blast wave theory. The projection of shock waves shown in the right of figures 17.1 and 17.2 is called Blast waves. These release large amount of energy instantaneously. In these implementations using the blast wave theory, we can get pressure distribution as well as shock wave shapes for the flat surface downstream of the blunt nose. The pressure distribution and the flow-field in the nose region cannot be provided by the blast wave theory.

Case I (Blunt nosed Slab)

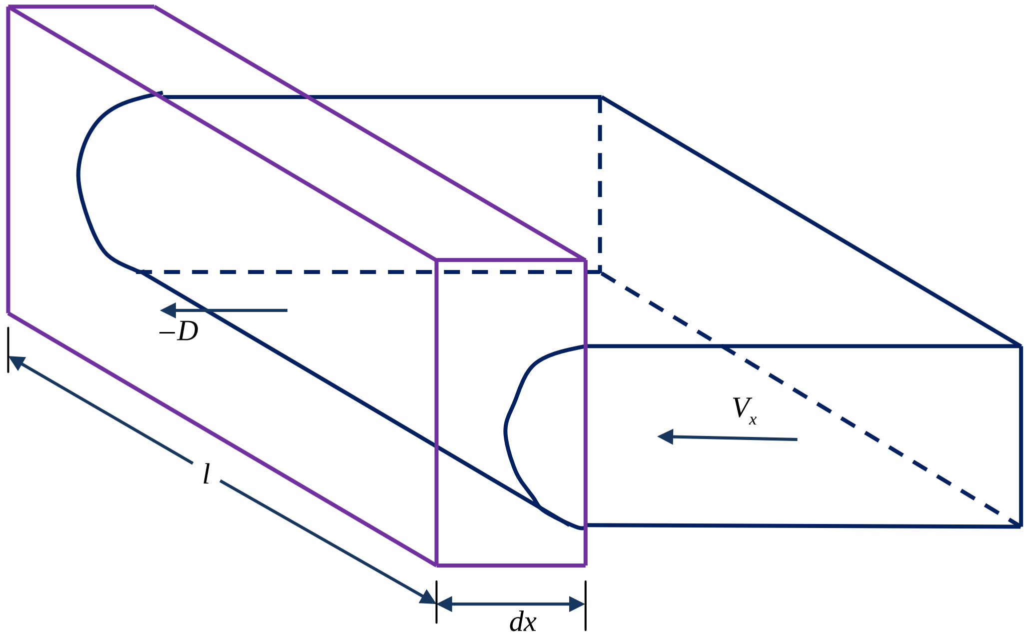

Fig. 17.3: Schematic of blunt slab for blast wave theory [1]

Consider the blunt nosed flat plate shown in Fig. 17.3. Let 'D' be the wave drag of the blunt nosed slab per unit span. Here wave drag should be inferred as the drag force offered to the object due to the high pressures behind the shock wave. As shown in the Fig. 17.3, the plate moves through a slab of air of thickness 'dx' in the direction of flight. Let us assume that the slab has unit span. Hence, for the motion in air, the body does work on the air which is equal to Ddx. Since work is form of energy, the amount of energy required for this motion per unit span is 'dE'. In view of blast wave theory, this can also be viewed as the energy released in unit span to generate the blast wave, Therefore,

$$dE = Ddx$$

If the slab is assumed to be moved by unit distance, then

$$E = D$$

Hence it can be said that the energy released per unit area is the drag force per unit span for the blunt nosed slab. In the unsteady one dimensional unsteady blast wave problem, in reference with the slab (Fig. 17.2), we have to assume a line of energy release which creates the blast waves normal to the plan of paper of infinite extent and propagating both upward and downward directions. The solution for pressure and shock location is readily available in the literature (Sedov, L. I. "Similarity and Dimensional Methods in Mechanics {translated from Russian by M. Friedmann, ed. M. Holl}, Academic Press, New York, 1959.).

For Planar Blast Wave,

$$p = k_2 \rho_\infty \left(\frac{E}{\rho_\infty} \right)^{\frac{2}{3}} t^{-2/3} \tag{17.1}$$

Where, $k_2 = \dfrac{2^{7/3}(2\gamma - 1)^{[(5\gamma-4)/3(2-\gamma)]}}{9(\gamma+1)^{[2(\gamma+1)/3(2-\gamma)]}}$

If ‘ r ’ is the co-ordinate of the shock wave, which is vertical co-ordinate of the wave

$$r = \left(\frac{E}{\rho_\infty} \right)^{1/3} t^{2/3} \tag{17.2}$$

As it can be seen here that time appears explicitly in the formulation, since this is the solution for unsteady propagation of blast wave. Here pressure ahead of the blast wave is neglected.

Case II (Blunt nosed Cylinder)

As in the case of blunt slab, the blunt nosed cylinder can be considered as moving the object through a cylindrical slab of thickness '*dx*'. If '*D*' is the wave drag for the blunt nosed cylinder then the energy required for this motion is,

$$dE = Ddx$$

Thus, for motion of unit length in the *x* direction for the cylinder,

$$E = D$$

Both these equations derived for cylindrical shock wave are same as that derived for blunt slab in context of planar blast wave. In view of the blast wave theory, we can consider a blast wave along an infinite line perpendicular to the page releasing energy E per unit length along this line. The blast wave or shock wave generated by this blast can be seen to be a cylindrically outward propagating shock wave. View of this wave will be a circle of increasing diameter in the plane of paper as shown in Fig. 17.1. The difference between the interpretations for slab and cylinder case are that, E is the energy released per unit area for blunt nosed slab and E is the energy released per unit length for blunt nosed cylinder. We can evaluate the pressure at the blast by neglecting the pressure ahead the blast wave as given in same reference mentioned for blunt slab.

$$p = k_1 \rho_\infty \left(\frac{E}{\rho_\infty} \right)^{\frac{1}{2}} t^{-1} \qquad (17.3)$$

Where, $k_1 = \dfrac{\gamma^{[2(\gamma-1)/(2-\gamma)]}}{2^{[(4-\gamma)/(2-\gamma)]}}$

If r is radial co-ordinate of the wave, then,

$$r = \left(\frac{E}{\rho_\infty} \right)^{1/4} t^{1/2} \qquad (17.4)$$

Lecture-18: Blast Wave Theory

18.1 Blast wave equivalence for blunt nosed slab

We have already analyzed the blast wave analogy for slab. This understanding can be extended for further evolution of the theory. We know that drag coefficient is,

$$C_D = \frac{D}{q_\infty S}$$

Where, D is the drag force

$$q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$$

And for unit span, area is, $S = d$ X1, where d=base height of the body

We have already derived the analog for drag that,

$$E = D = \frac{1}{2}\rho_\infty V_\infty^2 .d.C_D \qquad (18.1)$$

Time can be expressed as, $t = V_\infty / x$ where x is the distance from nose tip.

This expression for time can be used to along with Eq. 18.1 to redefine the pressure on the surface expressed by Eq. 17.1.

$$p = k_2 \rho_\infty \left[\frac{\frac{1}{2}\rho_\infty V_\infty^2 d C_D}{\rho_\infty} \right] \left(\frac{x}{V_\infty} \right)^{-2/3}$$

However we know that, for perfect gas,

$$\rho_\infty = \frac{p_\infty}{RT_\infty} = \frac{\gamma p_\infty}{\gamma RT_\infty} \text{ and}$$

$$\gamma RT_\infty = a_\infty^2 \text{ and } \frac{V_\infty^2}{a_\infty^2} = M_\infty^2,$$

$k_2 = 0.3438$ for air since $\gamma = 1.4$ for air

Therefore, the expression for pressure is,

$$\frac{p}{p_\infty} = 0.127 M_\infty^2 C_D^{\ 2/3} \left(\frac{x}{d}\right)^{-2/3} \tag{18.2}$$

Similarly we can calculate the shock location expressed by Eq. 17.2.

$$r = \left[\frac{1}{2} V_\infty^2 . C_D . d\right]^{1/3} \left(\frac{x}{V_\infty}\right)^{2/3}$$

$$\frac{r}{d} = \left(\frac{1}{2}\right)^{1/3} C_D^{\ 1/3} \left(\frac{x}{d}\right)^{2/3}$$

$$\frac{r}{d} = 0.793 C_D^{\ 1/3} \left(\frac{x}{d}\right)^{2/3} \tag{18.3}$$

The expression given by Eq. 18.2 and 18.3 provide nondimensional pressure and shock location for blunt slab using blast wave theory. We see here that, pressure distribution varies inversely with $x^{2/3}$ and the shock wave shape varies as $x^{2/3}$. The pressure value at a point in function of square of Mach number and depends on C_D. We can consider the $C_D = \frac{4}{3}$ for blunt nosed slab as given by Newtonian theory.

18.2. Blast wave equivalence for blunt nosed cylinder

If C_D is the drag coefficient for hypersonic flow over a blunt nosed cylinder, then

$$C_D = \frac{D}{q_\infty S}$$

Where, *D* is the drag force

$$q_\infty = \frac{1}{2} \rho_\infty V_\infty^2$$

And $S = \frac{\pi}{4} d^2$, where *d*=Base diameter of the body

But we have already evaluated the analogy between drag and energy for blast wave theory. Hence,

$$E = D = \frac{1}{2}\rho_\infty V_\infty^2 . C_D . \frac{\pi}{4} d^2$$

We can express the time as $t = \frac{x}{V_\infty}$

Using the equation for energy and time, the expression for pressure given by Eq. 17.3. becomes,

$$p = k_1 \rho_\infty \sqrt{\frac{\pi}{8}} V_\infty d \sqrt{C_D} \frac{V_\infty}{x}$$

Also for a perfect gas, $\rho_\infty = \frac{p_\infty}{RT_\infty} = \frac{\gamma p_\infty}{\gamma RT_\infty}$

Substituting this value of ρ_∞, we get,

$$p = k_1 \frac{\gamma p_\infty}{\gamma RT_\infty} \sqrt{\frac{\pi}{8}} V_\infty^2 \sqrt{C_D} \left(\frac{x}{d}\right)^{-1}$$

Now, $\gamma RT_\infty = a_\infty^2$ and $\frac{V_\infty^2}{a_\infty^2} = M_\infty^2$, also for air $\gamma = 1.4$, the equation for pressure transforms as,

$$\frac{p}{p_\infty} = 0.8773 k_1 M_\infty^2 \sqrt{C_D} \left(\frac{x}{d}\right)^{-1}$$

But, $k_1 = 0.07768$ for air since $\gamma = 1.4$, hence,

So, equation (66) becomes,

$$\frac{p}{p_\infty} = 0.0681 M_\infty^2 \frac{\sqrt{C_D}}{\left(\frac{x}{d}\right)} \qquad (18.4)$$

Similarly we can find out the shock location for blunt nosed cylinder starting with Eq. 17.4.

$$r = \left[\frac{1}{2} V_\infty^2 . C_D . \frac{\pi}{4} d^2 \right]^{1/4} \left(\frac{x}{V_\infty} \right)^{1/2}$$

$$\frac{r}{d} = \left(\frac{\pi}{8} \right)^{1/4} C_D^{\ 1/4} \sqrt{\frac{x}{d}}$$

$$\frac{r}{d} = 0.792 C_D^{\ 1/4} \sqrt{\frac{x}{d}} \qquad (18.5)$$

Equations 18.4 and 18.5 and are the outcome of blast wave theory for blunt nosed cylinders. We see here that, pressure distribution varies inversely with x and the shock wave shape varies as $x^{1/2}$. The pressure value at a point in function of square of Mach number and depends on C_D. We can consider the $C_D = 1$ for blunt nosed cylinder as given by Newtonian theory.

Lecture-19: Thin Shock Layer Theory

19.1 Thin Shock Layer Theory

Thin shock layer theory is based on the assumption that the shock is very much closer to the body which in turn leads to small volume between shock and body. This situation is typical of very high Mach number flows over generic hypersonic configurations. In such situations, we can assume that, $M_\infty \to \infty$ and $\gamma \to 1$. As it has been already observed that the shock angle and Mach angle are almost equal for hypersonic flow regime, we can express this fact as $\beta \to \theta$. For such high Mach condition within the shock layer, we will have same equation for shock, body and any streamline in the shock. This is the basic assumption of thin shock layer theory.

Consider the body and the shock as shown in Fig. 19.1. Here the co-ordinate system is such that x axis is parallel to the shock while y axis is perpendicular to the shock. Let u and v be the components of velocity in the x and y directions respectively. Let us assume the flow to be two-dimensional flow for the present illustration.

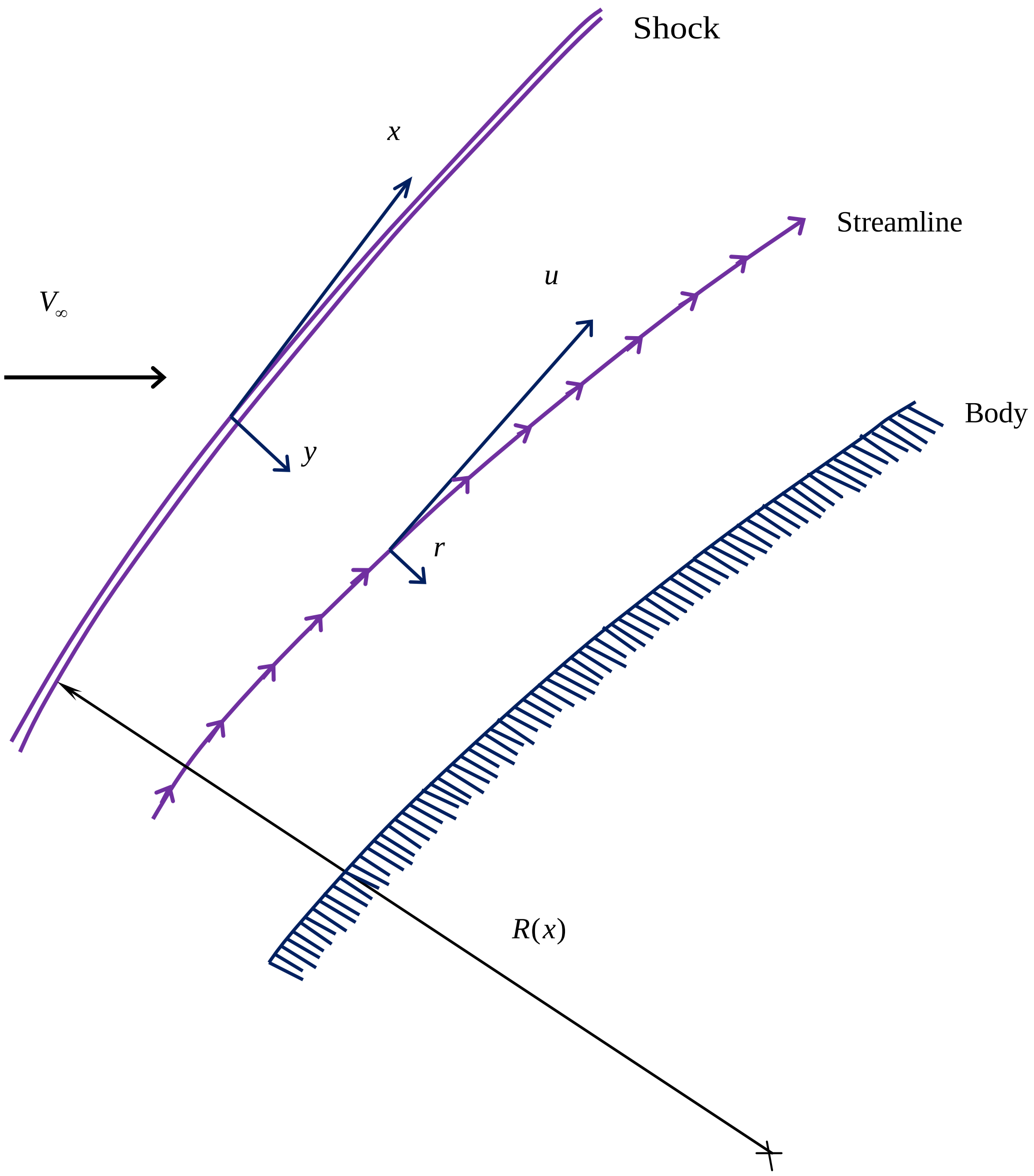

Fig. 19.1: Illustration for thin shock layer theory [1]

The momentum equation for the present coordinate system is

$$\rho \frac{u^2}{R} = \frac{\partial p}{\partial y}$$

Since our assumptions include thin shock layer and same equation for shock, streamlines and body. Here, R is the local streamline radius of curvature. For the thin shock-layer assumptions,

$$\rho \frac{u^2}{R_s} = \frac{\partial p}{\partial y} \tag{19.1}$$

Here R_s is the shock radius of curvature.

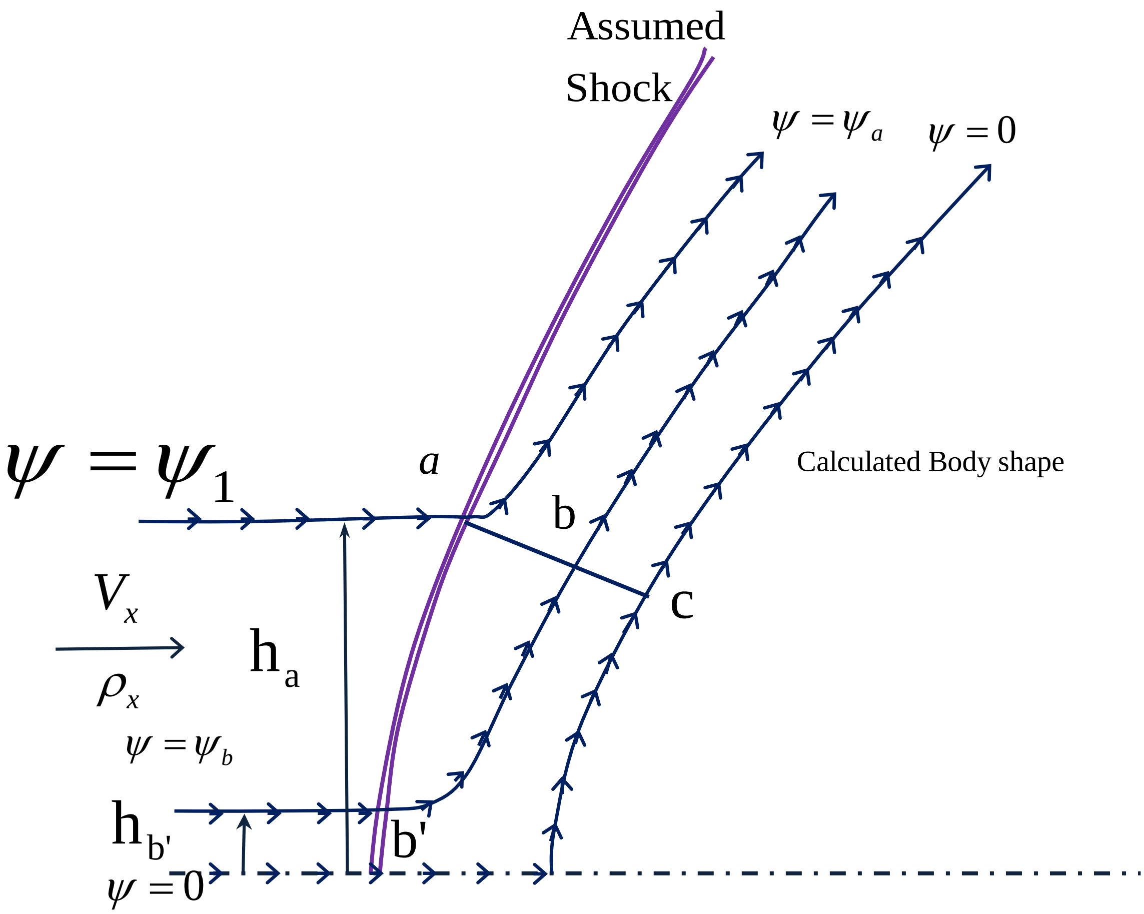

Fig 19.2: Body shape determination using thin shock layer theory [1]

Let ψ defines the streamline, such that,

$$\rho u = \frac{\partial \psi}{\partial y}$$

Replacing dy in equation (19.1) with this equation,

$$\frac{\rho u^2}{R_s} = \left(\frac{\partial p}{\partial \psi}\right)(\rho u)$$

$$\frac{\partial p}{\partial \psi} = \frac{u}{R_s} \tag{19.2}$$

Inline with the earlier assumptions made in thin shock layer, we can consider $u \approx u_s$ where u_s is the the velocity just behind the shock. By this assumption we are re-asserting the assumption that all the streamlines are parallel to the shock, Therefore,

$$\frac{\partial p}{\partial \psi} = \frac{u_s}{R_s} \tag{19.3}$$

We can integrate the Eq. (19.3) between a point in the shock layer where the value of the stream function as ψ and just behind the shock layer, where $\psi = \psi_s$.

$$p(x,\psi) = p_s(x) + \frac{u_s(x)}{R_s(x)}\left[\psi - \psi_s(x)\right] \tag{19.4}$$

Using the equation (19.4) we can build an inverse method where a shock wave shape will be assumed for a body to solve the above equation and then obtain shape and pressure distribution over the body. Thus obtained body shape when matches with the real shape then we can get the shock shape and pressure distribution. The procedure described by Maslen [Ref] can be summarized as,

1. Assume a shock wave shape for the known shape as shown in Fig. 19.2
2. This helps us to know all the flow quantities at point a (Fig. 19.2) just behind the shock using oblique shock relations. The value of $\psi = \psi_a$

   $$\psi_a = \rho_\infty V_\infty h_a$$

3. Consider any streamline of stream function ψ_b, where $0 < \psi_b < \psi_a$. This helps us to identify the point b inside the shock layer during the travel in y direction from a.
4. Calculate the pressure at point b from Eq 19.4.

   $$p_b = p_a + \frac{u_1}{R_s}(\psi_b - \psi_a)$$

5. Since the streamline passing through b and b' is same, we have $\psi_b = \psi_{b'}$.

   Such that ,

   $$\psi_{b'} = \psi_b = \rho_\infty V_\infty h_{b'}$$

6. Calculate all the thermodynamic properties at b from known pressure and entropy since entropy at b' is equal to entropy at b.

7. Calculate the velocity at point 2 from the adiabatic energy equation (total enthalpy is constant), that is,

$$h_0 = h_\infty + \frac{V_\infty^2}{2}$$

here, h_0 is the total enthalpy. In turn,

$$h_0 = h_\infty + \frac{u_b^2}{2} \text{ (Ignoring } v_2\text{)}$$

$$u_b = \sqrt{2(h_0 - h_b)}$$

8. Now all the flow quantities are known at point b. Repeat the same procedure at all the points between a and c where body surface is defined as $\psi = 0$.
9. The y co-ordinate, at a particular value of ψ can now be found by integrating the definition of the stream function. Since,

$$\frac{d\psi}{dy} = \rho u$$

Hence, $\int_{\psi}^{\psi_s} \frac{d\psi}{\rho u} = y$

Where, ρ and u are known and ψ is known from the previous steps. This also locates the body co-ordinate, where,

$$y_{body} = \int_0^{\psi_s} \frac{d\psi}{\rho u}$$

10. This procedure should be repeated for any desired number of points for generating desired body shape.

Reference:

Maslen, S. H. "Inviscid Hypersonic Flow Past Smooth Symmetric Bodies," AIAA Journal, vol. 2, no. 6, June 1964, pp. IO55-1O6I, as explained in Ref: Anderson, John D., Jr.: “Hypersonic and High Temperature Gas Dynamics”, McGraw-Hill Book Co., New York, 1989