Module-5: Hypersonic Boundary Layer theory

Lecture-20: Hypersonic boundary equation

20.1 Governing Equations for Viscous Flows

The Navier-Stokes (NS) equaadtions are the governing equations for the viscous compressible flow and hence are the governing equations for hypersonic flows. This section deals with the basics of NS equations and its non-dimensionalization.

Continuity Equation:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + V\nabla\rho$$

Considering steady state conditions we have;

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \text{ i.e. } \frac{\partial \rho}{\partial t} = 0 \qquad (20.1)$$

X Momentum Equation:

$\frac{D(\rho u)}{Dt} = \frac{\rho Du}{Dt} + \frac{uD\rho}{Dt}$, now since $\frac{D\rho}{Dt} = 0$;

$\frac{D(\rho u)}{Dt} = \frac{\rho Du}{Dt}$; therefore the L.H.S. simplifies to give X momentum equation in steady state conditions as;

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)\right] \qquad (20.2)$$

Y Momentum Equation:

$\frac{D(\rho v)}{Dt} = \frac{\rho Dv}{Dt} + \frac{vD\rho}{Dt}$, now since $\frac{D\rho}{Dt} = 0$;

$\frac{D(\rho v)}{Dt} = \frac{\rho Dv}{Dt}$; therefore the L.H.S. simplifies to give Y momentum equation in steady State Conditions as;

$$\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right)\right] \qquad (20.3)$$

Energy Equation:

$$\frac{D\left[\rho\left(e+\frac{(u^2+v^2)}{2}\right)\right]}{Dt}=\frac{\rho D\left(e+\frac{(u^2+v^2)}{2}\right)}{Dt}+\frac{\left(e+\frac{(u^2+v^2)}{2}\right)D\rho}{Dt},$$

Since $\frac{D\rho}{Dt}=0$;

$$\frac{D\left[\rho\left(e+\frac{(u^2+v^2)}{2}\right)\right]}{Dt}=\frac{\rho D\left(e+\frac{(u^2+v^2)}{2}\right)}{Dt},$$

Therefore the L.H.S. simplifies to give Energy Equation in steady State Conditions as;

$$\nabla\bullet\left(u\left(e+\frac{V^2}{2}\right)\right)=\rho\dot{q}+\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)-\left(\frac{\partial(pu)}{\partial x}+\frac{\partial(pv)}{\partial y}\right)+\tau_{xx}\frac{\partial u}{\partial x}+\tau_{xy}\frac{\partial u}{\partial y}+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y} \qquad (20.4)$$

The above equations are written for steady, compressible, viscous, two dimensional flows in Cartesian coordinates. Where u and v are velocities in x and y directions respectively; e represents internal energy per unit mass and $\dot{q}$ represents the volumetric heating that might occur. All other notations carry their usual meaning. We can simplify the above set of equations via appropriate assumptions, and obtain approximate viscous flow results.

20.2 Non Dimensional Form of Governing Equations

The non dimensional form of Navier-Stokes equations can be obtained as follows. Here we have considered a two dimensional steady flow and ignored the normal stresses τ_{xx} and τ_{yy}. Reference variables of the flow can be used for non-dimensionalization.

Non Dimensional Variables:

$$\bar{u} = \frac{u}{V_\infty} \qquad \bar{v} = \frac{v}{V_\infty} \qquad \bar{x} = \frac{x}{c} \qquad \bar{y} = \frac{y}{c} \qquad \bar{p} = \frac{p}{\rho V^2_\infty}$$

$$\bar{e} = \frac{e}{C_v T_\infty}$$

$$\bar{\mu} = \frac{\mu}{\mu_\infty} \qquad \bar{\kappa} = \frac{\kappa}{\kappa_\infty} \qquad \bar{\rho} = \frac{\rho}{\rho_\infty}$$

Where V_∞, T_∞, ρ_∞, μ_∞, κ_∞ are free stream parameters and c reference length.

Therefore the Non Dimensional Equations are given as:

Non Dimensional Continuity Equation:

$$\frac{\partial(\overline{\rho u})}{\partial \bar{x}} + \frac{\partial(\overline{\rho v})}{\partial \bar{y}} = 0 \tag{20.5}$$

Non Dimensional X Momentum Equation:

$$\overline{\rho u}\frac{\partial \bar{u}}{\partial \bar{x}} + \overline{\rho v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\gamma M^2_\infty}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial \bar{y}}\left[\bar{\mu}\left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}}\right)\right] \tag{20.6}$$

Non Dimensional Y Momentum Equation:

$$\overline{\rho u}\frac{\partial \bar{v}}{\partial \bar{x}} + \overline{\rho v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\gamma M^2_\infty}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial \bar{x}}\left[\bar{\mu}\left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}}\right)\right] \tag{20.7}$$

Non Dimensional Energy Equation:

$$\overline{\rho u}\frac{\partial \bar{e}}{\partial \bar{x}}+\overline{\rho v}\frac{\partial \bar{e}}{\partial \bar{y}}=\gamma(\gamma-1)M^2_\infty\left[\overline{\rho u}\frac{\partial}{\partial \bar{x}}\left(\bar{u}^2+\bar{v}^2\right)+\overline{\rho v}\frac{\partial}{\partial \bar{y}}\left(\bar{u}^2+\bar{v}^2\right)\right]+\frac{\gamma}{\Pr_\infty \mathrm{Re}_\infty}\left[\frac{\partial}{\partial \bar{x}}\left(\bar{\kappa}\frac{\partial \bar{T}}{\partial \bar{x}}\right)+\frac{\partial}{\partial \bar{y}}\left(\bar{\kappa}\frac{\partial \bar{T}}{\partial \bar{y}}\right)\right]$$

$$+\gamma(\gamma-1)\frac{M^2_\infty}{\mathrm{Re}_\infty}\left\{\frac{\partial}{\partial \bar{x}}\left[\overline{\mu v}\left(\frac{\partial \bar{v}}{\partial \bar{x}}+\frac{\partial \bar{u}}{\partial \bar{y}}\right)\right]+\frac{\partial}{\partial \bar{y}}\left[\overline{\mu u}\left(\frac{\partial \bar{v}}{\partial \bar{x}}+\frac{\partial \bar{u}}{\partial \bar{y}}\right)\right]\right\} \quad (20.8)$$

20.3 Process of Non-dimensionlisation of Governing Equations:-

Continuity Equation:-

We Have,

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0,$$

With non-dimensional parameters

$$\frac{\rho_\infty V_\infty}{c}.\left[\frac{\partial(\rho u/\rho_\infty V_\infty)}{\partial(x/c)}+\frac{\partial(\rho v/\rho_\infty V_\infty)}{\partial(y/c)}\right]=0$$

$$\frac{\partial(\overline{\rho u})}{\partial x}+\frac{\partial(\overline{\rho v})}{\partial y}=0$$

X & Y momentum Equations:-

The process of non dimensionlisation is almost the same for both X & Y momentum equations, therefore we have for X momentum equation;

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right)\right]$$

Rewriting the above Equation with Non Dimensional parameters, we get;

$$\frac{\rho_\infty V_\infty{}^2}{c}\left[\frac{\rho u}{\rho_\infty V_\infty}\frac{\partial(u/V_\infty)}{\partial(x/c)}+\frac{\rho u}{\rho_\infty V_\infty}\frac{\partial(v/V_\infty)}{\partial(y/c)}\right]=-\frac{p_\infty}{c}\frac{\partial(p/p_\infty)}{\partial(x/c)}$$

$$+\frac{\mu_\infty V_\infty}{c^2}\left\{\frac{\partial}{\partial(y/c)}\left[\frac{\mu}{\mu_\infty}\left(\frac{\partial(v/V_\infty)}{\partial(x/c)}+\frac{\partial(u/V_\infty)}{\partial(y/c)}\right)\right]+\frac{\partial}{\partial(x/c)}\left[\frac{\mu}{\mu_\infty}\left(\frac{\partial(v/V_\infty)}{\partial(x/c)}+\frac{\partial(u/V_\infty)}{\partial(y/c)}\right)\right]\right\}$$

Now dividing the above equation on both sides by the factor $\frac{\rho_\infty V_\infty^2}{c}$; we get term wise

First, the pressure term on Right Hand side (R.H.S); $\frac{p_\infty}{c}\frac{c}{\rho_\infty V_\infty^2} = \frac{p_\infty}{\rho_\infty V_\infty^2}$

Since Mach Number can be written as $M^2 = \frac{V_\infty^2}{a^2}$ therefore;

$\frac{p_\infty}{\rho_\infty V_\infty^2} = \frac{p_\infty}{\rho_\infty M^2 a^2}$ and furthermore, $p_\infty = \rho_\infty RT_\infty$ and $a^2 = \gamma RT_\infty$; the term reduces to

$$\frac{p_\infty}{\rho_\infty M^2 a^2} = \frac{RT_\infty}{M^2 \gamma RT_\infty} = \frac{1}{\gamma M^2}$$

Now the viscous term, $\frac{\mu_\infty V_\infty}{c^2}\frac{c}{\rho_\infty V_\infty^2} = \frac{\mu_\infty}{\rho_\infty V_\infty c} = \frac{1}{\mathrm{Re}_\infty}$

Thus the X momentum Equation reduces to,

$$\overline{\rho u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{\rho v}\frac{\partial \overline{u}}{\partial \overline{y}} = -\frac{1}{\gamma M^2_\infty}\frac{\partial \overline{p}}{\partial \overline{x}} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial \overline{y}}\left[\overline{\mu}\left(\frac{\partial \overline{v}}{\partial \overline{x}} + \frac{\partial \overline{u}}{\partial \overline{y}}\right)\right]$$

& similarly Y momentum Equation reduces to,

$$\overline{\rho u}\frac{\partial \overline{v}}{\partial \overline{x}} + \overline{\rho v}\frac{\partial \overline{v}}{\partial \overline{y}} = -\frac{1}{\gamma M^2_\infty}\frac{\partial \overline{p}}{\partial \overline{y}} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial \overline{x}}\left[\overline{\mu}\left(\frac{\partial \overline{v}}{\partial \overline{x}} + \frac{\partial \overline{u}}{\partial \overline{y}}\right)\right]$$

Normal stresses τ_{xx} and τ_{yy} are ignored for the sake of simplicity.

Lecture-21: Non-dimensionalisation of governing equations

21.1 Process of Non-dimensionlisation of Governing Equations

Energy Equation:

We know the energy equation as Eq. (20.4). Expanding the LHS of this equation, we get,

$$\rho u \frac{\partial}{\partial x}\left[e+\frac{(u^2+v^2)}{2}\right]+\rho v \frac{\partial}{\partial y}\left[e+\frac{(u^2+v^2)}{2}\right]=$$

$$\frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right)-\left(\frac{\partial (pu)}{\partial x}+\frac{\partial (pv)}{\partial y}\right)+\frac{\partial (u\tau_{xx})}{\partial x}+\frac{\partial (u\tau_{xy})}{\partial y}+\frac{\partial (v\tau_{xy})}{\partial x}+\frac{\partial (v\tau_{yy})}{\partial y}$$

Now simplifying the above equation by neglecting normal stresses τ_{xx} and τ_{yy}, we get

$$\rho u \frac{\partial e}{\partial x}+\rho v \frac{\partial e}{\partial y}=-\rho u \frac{\partial}{\partial x}\left[\frac{(u^2+v^2)}{2}\right]-\rho v \frac{\partial}{\partial y}\left[\frac{(u^2+v^2)}{2}\right]+$$

$$\frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right)-\left(\frac{\partial (pu)}{\partial x}+\frac{\partial (pv)}{\partial y}\right)+\frac{\partial}{\partial x}\left\{v\left(\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)\right\}+\frac{\partial}{\partial y}\left\{u\left(\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)\right\}$$

Now non dimensionalising Left hand side (L.H.S.) & R.H.S. terms separately;

L.H.S.=

$$\frac{\rho_\infty V_\infty C_v T_\infty}{c}\left[\frac{\rho u}{\rho_\infty V_\infty}\frac{\partial (e/C_v T_\infty)}{\partial (x/c)}+\frac{\rho v}{\rho_\infty V_\infty}\frac{\partial (e/C_v T_\infty)}{\partial (y/c)}\right]=\frac{\rho_\infty V_\infty C_v T_\infty}{c}\left[\overline{\rho u}\frac{\partial \bar{e}}{\partial \bar{x}}+\overline{\rho v}\frac{\partial \bar{e}}{\partial \bar{y}}\right]$$

Non-dimensional R.H.S., 1st & 2nd term;

$$\frac{\rho_\infty V_\infty^3}{c}\left\{\frac{\rho u}{\rho_\infty V_\infty}\frac{\partial}{\partial (x/c)}\left[\frac{(u/V_\infty)^2+(v/V_\infty)^2}{2}\right]+\frac{\rho v}{\rho_\infty V_\infty}\frac{\partial}{\partial (y/c)}\left[\frac{(u/V_\infty)^2+(v/V_\infty)^2}{2}\right]\right\}=$$

$$\frac{\rho_\infty V_\infty^3}{c}\left[\overline{\rho u}\frac{\partial (u^2+v^2)}{\partial \bar{x}}+\overline{\rho v}\frac{\partial (u^2+v^2)}{\partial \bar{y}}\right]$$

Non-dimensional R.H.S., 3rd & 4th term;

$$\frac{\kappa_\infty T_\infty}{c^2}\left[\frac{\partial}{\partial(x/c)}\left(\frac{\kappa}{\kappa_\infty}\frac{\partial(T/T_\infty)}{\partial(x/c)}\right)+\frac{\partial}{\partial(y/c)}\left(\frac{\kappa}{\kappa_\infty}\frac{\partial(T/T_\infty)}{\partial(y/c)}\right)\right]=\frac{\kappa_\infty T_\infty}{c^2}\left[\frac{\partial}{\partial\overline{x}}\left(\overline{\kappa}\frac{\partial\overline{T}}{\partial\overline{x}}\right)+\frac{\partial}{\partial\overline{y}}\left(\overline{\kappa}\frac{\partial\overline{T}}{\partial\overline{y}}\right)\right]$$

Non-dimensional R.H.S., 6th & 7th term reduces to;

$$\frac{\mu_\infty V_\infty^2}{c^2}\left\{\frac{\partial}{\partial(x/c)}\left\{\frac{v}{V_\infty}\left[\frac{\mu}{\mu_\infty}\left(\frac{\partial(v/V_\infty)}{\partial(x/c)}+\frac{\partial(u/V_\infty)}{\partial(y/c)}\right)\right]\right\}\right\}=\frac{\mu_\infty V_\infty^2}{c^2}\left\{\frac{\partial}{\partial\overline{x}}\left\{\overline{v}\left[\overline{\mu}\left(\frac{\partial\overline{v}}{\partial\overline{x}}+\frac{\partial\overline{u}}{\partial\overline{y}}\right)\right]\right\}\right\}$$

$$\frac{\mu_\infty V_\infty^2}{c^2}\left\{\frac{\partial}{\partial(y/c)}\left\{\frac{u}{V_\infty}\left[\frac{\mu}{\mu_\infty}\left(\frac{\partial(v/V_\infty)}{\partial(x/c)}+\frac{\partial(u/V_\infty)}{\partial(y/c)}\right)\right]\right\}\right\}=\frac{\mu_\infty V_\infty^2}{c^2}\left\{\frac{\partial}{\partial\overline{y}}\left\{\overline{u}\left[\overline{\mu}\left(\frac{\partial\overline{v}}{\partial\overline{x}}+\frac{\partial\overline{u}}{\partial\overline{y}}\right)\right]\right\}\right\}$$

Non-dimensional R.H.S., 5th term reduces to;

$$\frac{p_\infty V_\infty}{c}\left[\frac{\partial(p/p_\infty u/V_\infty)}{\partial(x/c)}\right]=\frac{p_\infty V_\infty}{c}\left[\frac{\partial(\overline{p}\overline{u})}{\partial\overline{x}}\right]$$

Now dividing the L.H.S. & R.H.S. by the factor $\dfrac{c}{\rho_\infty V_\infty C_v T_\infty}$, we obtain R.H.S. terms as;

For term 1st & 2nd;

$$\frac{\rho_\infty V_\infty^3}{c}\frac{c}{\rho_\infty V_\infty C_v T_\infty}=\frac{V_\infty^2}{C_v T_\infty}=\frac{V_\infty^2(\gamma-1)}{RT_\infty}=\frac{V_\infty^2\gamma(\gamma-1)}{a^2}=M_\infty^2\gamma(\gamma-1)$$

For term 3rd & 4th;

$$\frac{\kappa_\infty T_\infty}{c^2}\frac{c}{\rho_\infty V_\infty C_v T_\infty}=\frac{\kappa_\infty}{c\rho_\infty V_\infty C_v}=\frac{\mu_\infty}{c\rho_\infty V_\infty}\frac{\kappa_\infty\gamma}{\mu_\infty C_p}=\frac{\gamma}{\mathrm{Re}_\infty \mathrm{Pr}_\infty}$$

For term 5th;

$$\frac{p_\infty V_\infty}{c}\frac{c}{\rho_\infty V_\infty C_v T_\infty}=\frac{p_\infty}{\rho_\infty C_v T_\infty}=\frac{RT_\infty(\gamma-1)}{RT_\infty}=(\gamma-1)$$

For term 6th & 7th;

$$\frac{\mu_\infty V_\infty^2}{c^2}\frac{c}{\rho_\infty V_\infty C_v T_\infty}=\frac{\mu_\infty}{c\rho_\infty V_\infty}\frac{V_\infty^2}{C_v T_\infty}=\frac{1}{\mathrm{Re}_\infty}\frac{(\gamma-1)V_\infty^2}{RT_\infty}=\frac{1}{\mathrm{Re}_\infty}\frac{\gamma(\gamma-1)V_\infty^2}{a^2}=\frac{\gamma(\gamma-1)M_\infty^2}{\mathrm{Re}_\infty}$$

Finally the Non dimensional Energy equation is given as;

$$\overline{\rho}\overline{u}\frac{\partial\overline{e}}{\partial\overline{x}}+\overline{\rho}\overline{v}\frac{\partial\overline{e}}{\partial\overline{y}}=\gamma(\gamma-1)M_\infty^2\left[\overline{\rho}\overline{u}\frac{\partial}{\partial\overline{x}}\left(\overline{u}^2+\overline{v}^2\right)+\overline{\rho}\overline{v}\frac{\partial}{\partial\overline{y}}\left(\overline{u}^2+\overline{v}^2\right)\right]+\frac{\gamma}{\mathrm{Pr}_\infty\mathrm{Re}_\infty}\left[\frac{\partial}{\partial\overline{x}}\left(\overline{\kappa}\frac{\partial\overline{T}}{\partial\overline{x}}\right)+\frac{\partial}{\partial\overline{y}}\left(\overline{\kappa}\frac{\partial\overline{T}}{\partial\overline{y}}\right)\right]$$

$$+\gamma(\gamma-1)\frac{M_\infty^2}{\mathrm{Re}_\infty}\left\{\frac{\partial}{\partial\overline{x}}\left[\overline{\mu}\overline{v}\left(\frac{\partial\overline{v}}{\partial\overline{x}}+\frac{\partial\overline{u}}{\partial\overline{y}}\right)\right]+\frac{\partial}{\partial\overline{y}}\left[\overline{\mu}\overline{u}\left(\frac{\partial\overline{v}}{\partial\overline{x}}+\frac{\partial\overline{u}}{\partial\overline{y}}\right)\right]\right\}$$

The non-dimensional parameters of the Navier-Stokes equation are,

Ratio of specific heats: $\gamma=\frac{C_p}{C_v}$

Mach number: $M_\infty=\frac{V_\infty}{a_\infty}$

Reynolds number: $\mathrm{Re}=\frac{\rho_\infty V_\infty \mu_\infty}{c}$

Prandtl number: $\mathrm{Pr}=\frac{\mu_\infty C_p}{k_\infty}$

These four dimensionless parameters are called similarity parameters, and are very important in determining the nature of a given viscous flow problem. Thermodynamic properties, as reflected by γ, are important for any high speed flow problem. A combination of thermodynamics and flow kinetic energy can be found in M_∞, and it is known that,

$$M_\infty \sim \frac{\text{Flow Kinetic Energy}}{\text{Flow Internal Energy}}$$

For Reynolds number, we have

$$\mathrm{Re} \sim \frac{\text{Inertia Force}}{\text{Viscous Force}}$$

Prandtl number, appearing in the energy equation, has the physical interpretation as,

$$\mathrm{Pr} \sim \frac{\text{Frictional dissipation}}{\text{Thermal conduction}}$$

Here γ and Pr are the properties of gas while Re and M involve flow properties as well.

Lecture-22: Order of magnitude estimate

22.1 Boundary conditions

An important difference between inviscid and viscous flows can be seen explicitly in the boundary conditions at the wall. The usual boundary condition for an inviscid flow is no mass transfer through the wall which mathematically gets expressed as the normal component of velocity to be zero at the wall. This boundary condition is termed as “free slip along the wall”. This boundary condition gets added with the cancellation of tangential velocity at the wall due to the existence of friction. This boundary condition is termed as “no slip along the wall”. Therefore both the components of velocity become zero for viscous wall boundary condition, that is,

$$\text{Wall boundary condition:} \qquad u=v=0$$

If there is mass transfer at the wall, then we have to express the normal velocity at the wall as per the known mass flow rate to satisfy the mass conservation equation. However tangential component of velocity will still remain zero at the wall if wall has zero velocity while for conservation of momentum.

There are two types of boundary conditions related with the energy equation. In one of them, wall is treated with isothermal wall temperature where the known temperature is assigned at the wall as,

$$\text{Constant wall temperature boundary condition:} \qquad T=T_w$$

Here T_w is the specified wall temperature. For non uniform temperature distribution along the surface we have,

$$\text{Variable wall temperature boundary condition:} \qquad T=T_w(s)$$

here $T_w(s)$ is the specified wall temperature variation as a function of distance along the surface (s). This boundary condition is very much suitable for high conductivity wall materials so as to keep the wall at known constant temperature variation.

However in case of insulators, where thermal conductivity is very low, the wall temperature usually remains unknown. In such cases, wall heat flux is treated as zero or wall is treated as adiabatic wall. The mathematical representation of this boundary condition is,

Adiabatic wall boundary condition: $$\dot{q}_w = k\left(\frac{\partial T}{\partial n}\right) = 0$$

Here $\dot{q}_w$ is the wall heat flux. Moreover in some cased wall heat flux distribution can be apriorily known. Therefore $\dot{q}_w$ or the wall heat transfer rate should be specified as the boundary condition. This wall heat flux is dependent on temperature gradient normal to the wall in the gas immediately above the wall.

22.2 Application to boundary layer flow

Consider the boundary layer along a flat plate of length c as shown in Fig. 22.1.

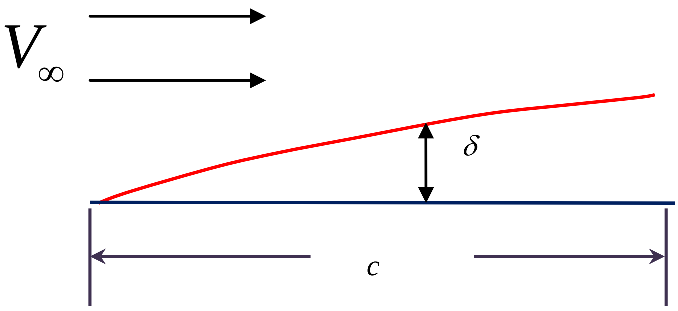

Fig. 22.1 Hypersonic flow over flat plate.

A thin layer of fluid is assumed to be decelerated in the presence of the wall. This assumption leads to the mathematical expression $\delta < c$. Here δ is the local boundary layer thickness. Apart from this, for hypersonic flow, we can also assume that, that are; $v << u$ and $\frac{\partial}{\partial x} << \frac{\partial}{\partial y}$

Now consider the Continuity Equation in Non Dimensional form;

$$\frac{\partial(\overline{\rho u})}{\partial x} + \frac{\partial(\overline{\rho v})}{\partial y} = 0$$

Here $\bar{u}$ varies from 0 at the wall to 1 at the edge of the boundary layer. Therefore we can consider that $\bar{u}$ has order of magnitude equal to 1. It is mathematically represented as O(1). On similar lines, we can as well mention for density as, $\bar{\rho}$ = O(1). Actually the x-coordinate of all the points in the fluid domain vary from 0 to c which is length of the plate. Therefore the non-dimensional x length scale can be represented as $\bar{x}$ =O(1). However the y co-ordinate of all the points at a particular x-location varies from 0 to δ where δ is the local boundary layer thickness. Hence the non-dimensional length scale $\bar{y}$ is smaller magnitude in comparison with other length scales. This can be represented as $\bar{y}$ =O(δ / c). For unit flat plate length, we have $\bar{y}$ =O(δ). Therefore from the continuity equation in terms of order of magnitude is,

$$\frac{[0(1)][0(1)]}{0(1)} + \frac{[0(1)][\bar{v}]}{0(\delta)} = 0$$

From the above equation it is clear that $\bar{v}$ must be of an order of magnitude equal to the local boundary layer thickness, δ , i.e. $\bar{v}$ =O(δ).

We know the non-dimensional form of X-momentum Eq. (20.6). Consider the order of magnitude form of each term as,

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} = 0(1) \qquad \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = 0(1) \qquad \frac{\partial \bar{p}}{\partial \bar{x}} = 0(1)$$

$$\frac{\partial}{\partial \bar{y}}\left(\bar{\mu}\frac{\partial \bar{v}}{\partial \bar{x}}\right) = 0(1) \qquad \frac{\partial}{\partial \bar{y}}\left(\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = 0\left(\frac{1}{\delta^2}\right)$$

Thus the order of magnitude equation for X momentum can be written as,

$$0(1) + 0(1) = -\frac{1}{\gamma M^2{}_\infty}0(1) + \frac{1}{\mathrm{Re}_\infty}\left[0(1) + \left(\frac{1}{\delta^2}\right)\right]$$

Lets assume that the Reynolds number is large. Therefore the term with Reynolds number in the denominator is of small magnitude which can be mathematically mentioned as,

$$\frac{1}{\mathrm{Re}_\infty} = 0(\delta^2)$$

Therefore the above equation now becomes;

$$0(1) + 0(1) = -\frac{1}{\gamma M^2{}_\infty} 0(1) + 0(\delta^2)\left[0(1) + \left(\frac{1}{\delta^2}\right)\right]$$

It is clear from this figure that product of $0(\delta^2)[0(1)] = 0(\delta^2)$ has very low order of magnitude in comparison with the rest of the terms in the same equation. This term actually is $\frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial \overline{y}}\left(\overline{\mu}\frac{\partial \overline{v}}{\partial \overline{x}}\right)$ in X momentum equation (Eq. 20.6). Since this term is very small in magnitude we can neglect it. Therefore the non-dimensional X-momentum equation can now be written as,

$$\overline{\rho u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{\rho v}\frac{\partial \overline{u}}{\partial \overline{y}} = -\frac{1}{\gamma M^2{}_\infty}\frac{\partial \overline{p}}{\partial \overline{x}} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial \overline{y}}\left(\overline{\mu}\frac{\partial \overline{u}}{\partial \overline{y}}\right) \tag{22.1}$$

The same in the dimensional form is,

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

This is X-momentum is valid for high Reynolds number flows having thin boundary layer at the wall.

Consider the Y momentum equation in non dimensional form given by Eq. (20.7);

$$\overline{\rho u}\frac{\partial \overline{v}}{\partial \overline{x}} + \overline{\rho v}\frac{\partial \overline{v}}{\partial \overline{y}} = -\frac{1}{\gamma M^2{}_\infty}\frac{\partial \overline{p}}{\partial \overline{y}} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial \overline{x}}\left[\overline{\mu}\left(\frac{\partial \overline{v}}{\partial \overline{x}} + \frac{\partial \overline{u}}{\partial \overline{y}}\right)\right]$$

Lets do the order of magnitude analysis of this equation.

$$0(\delta) + 0(\delta) = -\frac{1}{\gamma M^2{}_\infty}\frac{\partial \overline{p}}{\partial \overline{y}} + 0(\delta^2)\left[0(1) + \left(\frac{1}{\delta}\right)\right]$$

Here all the terms are of very small magnitude $0(\delta)$ except the term of pressure gradient if $\gamma M^2{}_\infty$ is of unity order of magnitude. Hence that term should also have very small order of magnitude. It implies that $\frac{\partial \overline{p}}{\partial \overline{y}} = 0(\delta)$. Therefore, from Y-momentum equation given by Eq. 20.7 gets transformed for the boundary layer theory as;

$$\frac{\partial p}{\partial y} = 0 \tag{22.2}$$

This equation clearly states that pressure at a particular X location does not change with Y- coordinate such that the gradient of pressure remains zero in the boundary layer. Therefore the pressure is only function of X co-ordinate, $p = p(x) = p_e(x)$, where $p_e(x)$ is the pressure distribution outside the boundary layer. However if freestream Mach number, M_∞, is very large so as to have $1/\gamma M^2{}_\infty = 0(\delta)$. In such cases, the pressure gradient in normal direction cab be large and still satisfy the Y-momentum equation. Hence for large Mach numbers, $\partial \overline{p} / \partial \overline{y}$ might be large enough to be expressed as $0(1)$. Hence pressure is not constant in the direction normal to the wall for hypersonic flows.

Lecture-23: Boundary layer equations

23.1 Boundary layer equations

Let's derive the energy equation under the assumption of very thin boundary layer for very high Reynolds number hypersonic flows. We know that the non-dimensional energy equation is given by Eq. 20.8. This energy equation is for total energy which means the summation of kinetic energy and internal energy. We have neglected the potential energy of the fluid particle. Therefore let us derive the energy equation for kinetic energy alone. Consider the X-momentum and Y-momentum equations given by Eq. (20.2) and (20.3). in the dimensional form. Multiplying the X momentum equation by u velocity, we get;

$$\rho u \frac{Du}{Dt} = \rho \frac{D(u^2/2)}{Dt} = -u\frac{\partial p}{\partial x} + u\frac{\partial \tau_{xx}}{\partial x} + u\frac{\partial \tau_{yx}}{\partial x}$$

Similarly multiply Y momentum equation by v velocity, we get;

$$\rho v \frac{Dv}{Dt} = \rho \frac{D(v^2/2)}{Dt} = -v\frac{\partial p}{\partial y} + v\frac{\partial \tau_{xy}}{\partial y} + v\frac{\partial \tau_{yy}}{\partial y}$$

Adding the above both the equations we have;

$$\rho \frac{D(u^2/2 + v^2/2)}{Dt} = \rho \frac{D(V^2/2)}{Dt} = -u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y} + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial x}\right) + v\left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

(23.1)

But the energy equation given by Eq. (20.4),

$$\rho u \frac{\partial}{\partial x}\left[e + \frac{(u^2+v^2)}{2}\right] + \rho v \frac{\partial}{\partial y}\left[e + \frac{(u^2+v^2)}{2}\right] = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right)$$

$$-\left(\frac{\partial (pu)}{\partial x} + \frac{\partial (pv)}{\partial y}\right) + \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{xy})}{\partial y} + \frac{\partial (v\tau_{xy})}{\partial x} + \frac{\partial (v\tau_{yy})}{\partial y}$$

Replacing the L.H.S. in terms of the substantial derivative form we get,

$$\rho u\frac{\partial}{\partial x}\left[e+\frac{V^2}{2}\right]+\rho v\frac{\partial}{\partial y}\left[e+\frac{V^2}{2}\right]=\rho\frac{D(e+V^2/2)}{Dt}=R.H.S. \qquad (23.2)$$

Subtracting Eq. (23.1) from (23.2), we get, the heat energy equation is;

$$\rho\frac{D(e)}{Dt}=\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)-\left(\frac{\partial(pu)}{\partial x}+\frac{\partial(pv)}{\partial y}\right)$$

$$+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{xy})}{\partial y}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}-u\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial x}\right)-v\left(\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{yy}}{\partial y}\right)$$

Let’s simplify the L.H.S.,

$$L.H.S.=\rho\frac{D(e)}{Dt}=\rho u\frac{\partial e}{\partial x}+\rho v\frac{\partial e}{\partial y}$$

From the definition of enthalpy we have,

$$e=h-p/\rho$$

Therefore the heat energy equation is,

$$\rho u\frac{\partial(h-p/\rho)}{\partial x}+\rho v\frac{\partial(h-p/\rho)}{\partial y}=\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-\rho u\frac{\partial p/\rho}{\partial x}-\rho v\frac{\partial p/\rho}{\partial y},$$

Further simplification of this equation leads to,

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-\rho u\frac{\partial p/\rho}{\partial x}-\rho v\frac{\partial p/\rho}{\partial y}=\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-\frac{\partial(pu)}{\partial x}-\frac{\partial(pv)}{\partial y}+p\rho\left(\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}\right)$$

Cancellation of similar terms from both the sides we have,

L.H.S.= $\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}$

Now, R.H.S. = $\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{xy})}{\partial y}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}$

$$+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}-u\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial x}\right)-v\left(\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{yy}}{\partial y}\right)$$

Simplifying the viscous terms,

$$\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{xy})}{\partial y}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}-u\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial x}\right)-v\left(\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{yy}}{\partial y}\right)=\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{yx}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{yx}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]$$

Therefore we have heat energy equation as,

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}$$

$$+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]$$

Now from Y momentum Boundary Layer Equation we have, $\frac{\partial p}{\partial y}=0$. Let us carry out the order of magnitude analysis for the above Energy Equation;

$$0(1)+0(1)=0(1)+\frac{1}{0(\delta^2)}+0(1)+\lambda\left(0(1)+0(1)\right)^2+\mu\left[2*0(1)+2*0(1)+\left(\frac{1}{\delta}+\delta\right)^2\right]$$

From the above equation it is very much clear that temperature gradient along Y axis has larger magnitude as compared to the temperature gradient along X axis. Moreover, we know from the momentum equation, that, $v<<u$ & $\frac{\partial}{\partial x}<<\frac{\partial}{\partial y}$ for boundary layer assumption. Therefore gradients of lower magnitude, Energy Equation can be expressed as

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+u\frac{dp_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{23.3}$$

The mass, X-momentum, Y-momentum and energy equations for the boundary layer are respectively as follows,

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial p}{\partial y}=0$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+u\frac{dp_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

These equations are non-linear. However assumptions of boundary layer theory make solution procedure simpler. Apart from this the pressure is only function of X-coordinate hence can be represented using an ordinary differential equation rather than a partial differential equation. The variables u, v, p, ρ, T and h are the unknowns in these equations. However, p can be known from $p=p_e(x)$. Rest of the variables likeμ and κ are properties of fluid and are temperature dependant. The following perfect gas relations should also be used to complete the set of equations.

$$p=\rho RT$$
$$h=C_pT$$

Boundary conditions to be considered to solve above equations are

At Wall: y=0, u=0, v=0, $T=T_w$ or $\left(\frac{\partial T}{\partial n}\right)_w=0$ (adiabatic wall)

At boundary layer edge: $y\rightarrow\infty$, $u\rightarrow u_e$, $T\rightarrow T_e$

Here subscript e stands for the values measured or known at the edge of the boundary layer. The boundary layer equations are valid for compressible subsonic or supersonic flow. In case of application for hypersonic flows, it should again be noted that the Y-momentum equation $\partial p/\partial y=0$ should be changed for high Mach number cases.

Lecture-24: Similarity solution for boundary layer equation

24.1 Similarity solution of compressible boundary layer equation.

Boundary layer theory makes it convenient to reduce the complexity of basic governing equation which has been derived using the order of magnitude estimate for the non-dimensional form of the governing equations. Hence following are the mass, momentum and energy equation which are to be solved for compressible boundary layer.

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0 \tag{24.1}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{24.2}$$

$$\frac{\partial p}{\partial y}=0 \tag{24.3}$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+u\frac{dp_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{24.4}$$

These equations are derived for the X-Y coordinate system. Hence the variation of all the properties is assumed to be dependant on X and Y co-ordinates. However, lets transform the dependence of all the variables from X and Y to new dependant variables (ξ,η). This transformation ensures the self similar solution for the velocity profile where $u=u(\eta)$ and independent of ξ. This transformation has the following dependant variables as,

$$\xi=\int_0^x \rho_e u_e \mu_e dx \tag{24.1}$$

$$\eta=\frac{u_e}{\sqrt{2\xi}}\int_0^y \rho dy \tag{24.2}$$

Here ρ_e, u_e and μ_e are the density, velocity and viscosity coefficients at the edge of the boundary layer and are functions of x only. Therefore $\xi = \xi(x)$.

These special variables chosen for transformation should be implemented for the governing equations derived especially for the boundary layer. This would lead to new form of the same governing equations. Following are the basic steps involved in this transformation.

Step 1. Replacement of derivatives of independent variables.

The new independent variables are expressed in terms of the old independent variables using Eq. (24.1) and (24.2). Now we have to express the derivatives of them in terms of new independent variables.

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right) \tag{24.3}$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right) \tag{24.4}$$

From the definition of $\xi = \xi(x)$ given by Eq. (24.1) we can write,

$$\frac{\partial \xi}{\partial x} = \rho_e u_e \mu_e \tag{24.5}$$

$$\frac{\partial \xi}{\partial y} = 0 \tag{24.6}$$

From the definition of η given by Eq. (24.2) we can write,

$$\frac{\partial \eta}{\partial y} = \frac{u_e \rho}{\sqrt{2\xi}} \frac{\partial}{\partial \eta} \tag{24.7}$$

Substituting Eqs. (24.5)-(24.7) into Eqs. (24.3) and (24.4), we can the derivatives as,

$$\frac{\partial}{\partial x} = \rho_e u_e \mu_e \frac{\partial}{\partial \xi} + \left(\frac{\partial \eta}{\partial x}\right)\frac{\partial}{\partial \eta} \tag{24.8}$$

$$\frac{\partial}{\partial y} = \frac{u_e \rho}{\sqrt{2\xi}} \frac{\partial}{\partial \eta} \tag{24.9}$$

We know the definition of the stream function ψ defined as

$$\frac{\partial\psi}{\partial y} = \rho u \qquad (24.10)$$

$$\frac{\partial\psi}{\partial x} = -\rho v \qquad (24.11)$$

The X momentum boundary layer equation given by Eq. (24.2) in terms of ψ is,

$$\frac{\partial\psi}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad (24.12)$$

Let’s introduce the derivatives from eq. (24.8) and (24.9) in the eq. (24.12) we get,

$$\left(\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial\psi}{\partial\eta}\right)\left[\rho_e u_e\mu_e\frac{\partial u}{\partial\xi} + \left(\frac{\partial\eta}{\partial x}\right)\frac{\partial u}{\partial\eta}\right] - \left[\rho_e u_e\mu_e\frac{\partial\psi}{\partial\xi} + \left(\frac{\partial\eta}{\partial x}\right)\frac{\partial\psi}{\partial\eta}\right]\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial u}{\partial\eta}$$

$$= -\rho_e u_e\mu_e\frac{dp_e}{d\xi} + \frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial}{\partial\eta}\left(\frac{u_e\rho\mu}{\sqrt{2\xi}}\frac{\partial u}{\partial\eta}\right) \qquad (24.13)$$

Multiplying Eq. (24.13) by $\sqrt{2\xi}\,/\,u_e\rho$, we get,

$$\frac{\partial\psi}{\partial\eta}\left[\rho_e u_e\mu_e\frac{\partial u}{\partial\xi} + \left(\frac{\partial\eta}{\partial x}\right)\frac{\partial u}{\partial\eta}\right] - \left[\rho_e u_e\mu_e\frac{\partial\psi}{\partial\xi} + \left(\frac{\partial\eta}{\partial x}\right)\frac{\partial\psi}{\partial\eta}\right]\frac{\partial u}{\partial\eta}$$

$$= -\sqrt{2\xi}\,\frac{\rho_e}{\rho}\mu_e\frac{dp_e}{d\xi} + \frac{\partial}{\partial\eta}\left(\frac{u_e\rho\mu}{\sqrt{2\xi}}\frac{\partial u}{\partial\eta}\right) \qquad (24.14)$$

Step 2. Replacement of dependent variables.

Let us define a function ‘f’ of ξ and η, $f(\xi,\eta)$, such that

$$\frac{u}{u_e} = \frac{\partial f}{\partial\eta} \equiv f' \qquad (24.15)$$

Here prime denotes the partial derivative of f with respect to η. We know that the velocity at the boundary layer edge is function of X alone. Hence it is function of ξ only ($u_e = u_e(\xi)$). Therefore we can get the following

$$\frac{\partial u}{\partial \xi} = f' \frac{du_e}{d\xi} + u_e \frac{\partial f'}{\partial \xi} \tag{24.16}$$

$$\frac{\partial u}{\partial \eta} = u_e f'' \tag{24.17}$$

These two steps make it easy to express the governing equations of the thin boundary layer in terms of new variables so as to make it easier to solve.

Lecture-25: Transformation of boundary layer equations

25.1 Transformation of boundary layer momentum equation.

We have already seen the two steps involved in getting simpler boundary layer equation. The steps followed to those are seen here.

Step 3. Expression of f in terms of ψ.

The stream function ψ has been introduced to reduce the number of dependant variables of the governing equations. Therefore let us express the new dependent variable $f(\xi,\eta)$ in terms of stream function. From eq. (24.9), (24.10) and (24.15) we can write the following,

$$\frac{u_e \rho}{\sqrt{2\xi}}\frac{\partial \psi}{\partial \eta} = \rho u = \rho f' u_e \text{ or } \frac{\partial \psi}{\partial \eta} = \sqrt{2\xi}\, f' \tag{25.1}$$

Integrating this equation with respect to η, we get,

$$\psi = \sqrt{2\xi}\, f + F(\xi) \tag{25.2}$$

Here $F(\xi)$ is any arbitrary function of ξ. However, from the definition of stream function, we know that difference in stream function casts mass flow rate. Therefore for the stream function should be anchored to zero at the non-blowing wall, $\psi(\xi,0)=0$. Hence $f=0$ and $F(\xi)=0$ ensure the zero value of ψ at the wall. This makes it clear that the any arbitrary function, represented by $F(\xi)$ must be zero, which leads to,

$$\psi = \sqrt{2\xi}\, f \tag{25.3}$$

We will also have,

$$\frac{\partial \psi}{\partial \xi} = \sqrt{2\xi}\frac{\partial f}{\partial \xi} + \frac{1}{\sqrt{2\xi}} f \tag{25.4}$$

Step 4. Derivation for final equation.

We can derive the final expression using Eqs. (24.16)- (24.17) and (25.1) and substituting into Eq. (24.14) which is X-momentum boundary layer equation, we get,

$$\sqrt{2\xi}\, f'\left[\rho_e u_e \mu_e\left(f'\frac{du_e}{d\xi}+u_e\frac{\partial f'}{\partial \xi}\right)+\left(\frac{\partial \eta}{\partial x}\right)u_e f''\right]$$

$$-\left[\rho_e u_e \mu_e\left(\sqrt{2\xi}\frac{\partial f}{\partial \xi}+\frac{1}{\sqrt{2\xi}}f\right)+\left(\frac{\partial \eta}{\partial x}\right)\sqrt{2\xi}\, f'\right]u_e f''$$

$$=-\sqrt{2\xi}\frac{\rho_e}{\rho}\mu_e\frac{dp_e}{d\xi}+\frac{\partial}{\partial \eta}\left(\frac{u_e^2\rho\mu}{\sqrt{2\xi}}f''\right) \qquad (25.5)$$

The Euler equation for the outer flow is,

$$dp_e = -\rho_e\, u_e du_e \qquad (25.6)$$

Using Eq. (25.6) we can get the Eq. (25.5) as

$$\sqrt{2\xi}\rho_e u_e^2\mu_e(f')^2\frac{\partial u_e}{\partial \xi}+\sqrt{2\xi}\, f'\rho_e u_e^2\mu_e\frac{\partial f'}{\partial \xi}+\sqrt{2\xi}\mu_e f f''\left(\frac{\partial \eta}{\partial x}\right)=$$

$$\sqrt{2\xi}\rho_e u_e^2\mu_e f''\frac{\partial f}{\partial \xi}-\frac{\rho_e u_e^2\mu_e}{\sqrt{2\xi}}ff''-\sqrt{2\xi}\mu_e f f''\left(\frac{\partial \eta}{\partial x}\right)$$

Therefore,

$$\sqrt{2\xi}\rho_e u_e^2\mu_e(f')^2\frac{\partial u_e}{\partial \xi}+\sqrt{2\xi}\, f'\rho_e u_e^2\mu_e\frac{\partial f'}{\partial \xi} \quad =\sqrt{2\xi}\frac{(\rho_e)^2}{\rho}u_e^2\mu_e\frac{du_e}{d\xi}+\frac{\partial}{\partial \eta}\left(\frac{u_e^2\rho\mu}{\sqrt{2\xi}}f''\right) \qquad (25.7)$$

It can be seen that the term involving $\partial\eta/\partial x$ does not appear in the Eq. (25.7). This the main reason for non evaluation of $\partial\eta/\partial x$ earlier explicitly. Simplified form of the same equation is,

$$\frac{1}{u_e}(f')^2\frac{du_e}{d\xi}+f'\frac{\partial f'}{\partial\xi}-f''\frac{\partial f}{\partial\xi}=\frac{\rho_e}{\rho}\frac{1}{u_e}\frac{du_e}{d\xi}+\frac{\partial}{\partial\eta}\left(\frac{1}{2\xi}\frac{\rho\mu}{\rho_e\mu_e}\right)f'' \tag{25.8}$$

Let's introduce a variable, $C=\rho\mu/\rho_e\mu_e$ and obtain the final form of the X-momentum equation in the transformed state.

$$\boxed{(Cf'')'+ff''=\frac{2\xi}{u_e}\left[(f')^2-\frac{\rho_e}{\rho}\right]\frac{du_e}{d\xi}+2\xi\left(f'\frac{\partial f'}{\partial\xi}-\frac{\partial f}{\partial\xi}f''\right)} \tag{25.9}$$

This transformed equation is valid for steady compressible flow in the thin boundary layer.

The Y-momentum boundary layer y-momentum equation, gets transformed as,

$$\boxed{\frac{\partial p}{\partial\eta}=0} \tag{25.10}$$

25.2 Transformation of boundary layer energy equation

We can obtain the energy equation in the transformed variables using same strategy incorporated earlier. Let's substitute Eq. (24.8)- (24.11) in the L.H.S. of Eq. (24.4), we obtain the transformed R.H.S. term of energy equation as

$$\frac{\partial\psi}{\partial y}\frac{\partial h}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+u\frac{dp_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{25.11}$$

Using Eq.(24.8), we can express the first term on LHS as,

$$\frac{\partial\psi}{\partial y}\frac{\partial h}{\partial x}=\left(\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial\psi}{\partial\eta}\right)\left(\rho_e u_e\mu_e\frac{\partial h}{\partial\xi}+\left(\frac{\partial\eta}{\partial x}\right)\frac{\partial h}{\partial\eta}\right) \tag{25.12}$$

Let's define the static enthalpy as the non dimensional variable,

$$g = g(\xi,\eta) = \frac{h}{h_e} \qquad (25.13)$$

Hence we can write as,

$$\frac{\partial h}{\partial \xi} = h_e \frac{\partial g}{\partial \xi} + g\,\frac{\partial h_e}{\partial \xi} \qquad (25.14)$$

$$\frac{\partial h}{\partial \eta} = h_e g' \qquad (25.15)$$

Here $g' = \partial g / \partial \eta$, since $h_e = h_e(x)$ hence its derivation with respect to η is zero.

Therefore using Eq. (25.14), (25.15), we can re-express the first term of LHS of Eq. (25.12) as,

$$\frac{\partial \psi}{\partial y}\frac{\partial h}{\partial x} = \left(\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial \psi}{\partial \eta}\right)\left(\rho_e u_e \mu_e \frac{\partial h}{\partial \xi} + \left(\frac{\partial \eta}{\partial x}\right)\frac{\partial h}{\partial \eta}\right)$$

$$= \left[\frac{u_e\rho}{\sqrt{2\xi}}\sqrt{2\xi}\, f'\right]\left[\rho_e u_e \mu_e \left(h_e\frac{\partial g}{\partial \xi} + g\,\frac{\partial h_e}{\partial \xi}\right) + \left(\frac{\partial \eta}{\partial x}\right) h_e g'\right]$$

$$= \left(u_e\rho\, f'\right)\left(\rho_e u_e \mu_e h_e \frac{\partial g}{\partial \xi} + \rho_e u_e \mu_e g\,\frac{\partial h_e}{\partial \xi} + \left(\frac{\partial \eta}{\partial x}\right) h_e g'\right)$$

$$\frac{\partial \psi}{\partial y}\frac{\partial h}{\partial x} = \rho_e \rho u_e^2 \mu_e h_e f'\,\frac{\partial g}{\partial \xi} + \rho_e \rho u_e^2 \mu_e f' g\,\frac{\partial h_e}{\partial \xi} + u_e\rho\, f'\left(\frac{\partial \eta}{\partial x}\right) h_e g' \qquad (25.16)$$

Consider the second term of L.HS. of Eq.(25.12) using Eqs. (24.8) and (24.9) in the same way, we get,

$$\frac{\partial \psi}{\partial x}\frac{\partial h}{\partial y} = \left(\rho_e u_e \mu_e \frac{\partial \psi}{\partial \xi} + \left(\frac{\partial \eta}{\partial x}\right)\frac{\partial \psi}{\partial \eta}\right)\left(\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial h}{\partial \eta}\right) \qquad (25.17)$$

Now substituting Eq.(25.15),(25.1) and (25.4) in Eq. (25.17), we have

$$\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial y}=\left[\rho_e u_e \mu_e\left(\sqrt{2\xi}\,\frac{\partial f}{\partial\xi}+\frac{1}{\sqrt{2\xi}}\,f\right)+\left(\frac{\partial\eta}{\partial x}\right)\sqrt{2\xi}\,f'\right]\left(\frac{u_e\rho}{\sqrt{2\xi}}h_e g'\right)$$

$$\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial y}=\rho_e\rho u_e^2\mu_e h_e g'\frac{\partial f}{\partial\xi}+\frac{\rho_e\rho u_e^2\mu_e h_e f g'}{2\xi}+\rho u_e h_e g' f'\left(\frac{\partial\eta}{\partial x}\right) \tag{25.18}$$

Therefore complete L.H.S. of Eq.(25.11) using Eq. (25.16) and (25.18) is

$$L.H.S.=\frac{\partial\psi}{\partial y}\frac{\partial h}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial y}=\rho_e\rho u_e^2\mu_e h_e f'\frac{\partial g}{\partial\xi}+\rho_e\rho u_e^2\mu_e f' g\frac{\partial h_e}{\partial\xi}+u_e\rho f'\left(\frac{\partial\eta}{\partial x}\right)h_e g'$$

$$-\rho_e\rho u_e^2\mu_e h_e g'\frac{\partial f}{\partial\xi}-\frac{\rho_e\rho u_e^2\mu_e h_e f g'}{2\xi}-\rho u_e h_e g' f'\left(\frac{\partial\eta}{\partial x}\right)$$

$$L.H.S.=\rho_e\rho u_e^2\mu_e\left[h_e f'\frac{\partial g}{\partial\xi}+f' g\frac{\partial h_e}{\partial\xi}-h_e g'\frac{\partial f}{\partial\xi}-\frac{h_e f g'}{2\xi}\right] \tag{25.19}$$

Lecture-26: Similarity solution of boundary layer equation

26.1 Transformation of boundary layer energy equation and similarity solution

The simpler forms of X-momentum and Y-momentum have been obtained. The energy equation has also been obtained in the transformed form for LHS. Let's consider the RHS of Eq. (25.11) for the transformation. RHS of the Eq. (25.11) is as.

$$R.H.S.=\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+u\frac{dp_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad (26.1)$$

Lets consider the first term of this RHS and express the temperature in terms of enthalpy as $T=h/C_p$ we get,

$$\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)=\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial}{\partial\eta}\left[\frac{\kappa u_e\rho}{\sqrt{2\xi}}\frac{\partial}{\partial\eta}\left(\frac{h}{C_p}\right)\right] \qquad (26.2)$$

If we consider the gas to be calorically perfect and variation of C_p to be negligible then, we get

$$\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)=\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial}{\partial\eta}\left[\frac{\rho\kappa u_e}{\sqrt{2\xi}}\frac{\partial}{\partial\eta}\left(\frac{h}{C_p}\right)\right]=\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial}{\partial\eta}\left[\frac{\rho\kappa u_e}{\sqrt{2\xi}C_p}h_e g'\right]$$

$$\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)=\frac{h_e u_e^2\rho}{2\xi}\frac{\partial}{\partial\eta}\left[\frac{\rho\mu}{\mathrm{Pr}}g'\right] \qquad (26.3)$$

For the 2nd term of Eq.(26.1), we know,

$$u\frac{dp_e}{dx}=u_e f'\left[\rho_e u_e\mu_e\frac{dp_e}{d\xi}\right]$$

$$u\frac{dp_e}{dx}=u_e f'\left[\rho_e u_e\mu_e\frac{dp_e}{d\xi}\right]=\rho_e u_e^2\mu_e f'\left[-\rho_e u_e\frac{du_e}{d\xi}\right]=-\rho_e^2 u_e^3\mu_e f'\frac{du_e}{d\xi} \qquad (26.4)$$

For the 3rd term, from Eq.(26.1), we get

$$\mu\left(\frac{\partial u}{\partial y}\right)^2=\mu\left[\frac{u_e\rho}{\sqrt{2\xi}}\frac{\partial u}{\partial \xi}\right]^2=\frac{u_e^2\rho^2\mu}{2\xi}\left(u_e f''\right)^2=\frac{u_e^4\rho^2\mu}{2\xi}\left(f''\right)^2 \tag{26.5}$$

Therefore the complete form of RHS of Eq. (26.1),

$$R.H.S.=\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+u\frac{dp_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2=\frac{h_e u_e^2\rho}{2\xi}\frac{\partial}{\partial \eta}\left[\frac{\rho\mu}{\Pr}g'\right]-\rho_e^2 u_e^3\mu_e f'\frac{du_e}{d\xi}+\frac{u_e^4\rho^2\mu}{2\xi}\left(f''\right)^2 \tag{26.6}$$

Therefore the complete form of Eq. (26.1) using Eq. (25.19) and (26.3)-(26.5) is

Dividing both sides by $\rho_e\rho u_e^2\mu_e$, we get

$$h_e f'\frac{\partial g}{\partial \xi}+f'g\frac{\partial h_e}{\partial \xi}-h_e g'\frac{\partial f}{\partial \xi}-\frac{h_e f g'}{2\xi}=\frac{h_e}{2\xi\rho_e\mu_e}\frac{\partial}{\partial \eta}\left[\frac{\rho\mu}{\Pr}g'\right]-\frac{\rho_e u_e f'}{\rho}\frac{du_e}{d\xi}+\frac{u_e^2\rho\mu}{2\xi\rho_e\mu_e}\left(f''\right)^2$$

Multiplying both sides by $2\xi/h_e$, introducing the term $C=\rho\mu/\rho_e\mu_e$ the above equation simplifies to,

$$2\xi f'\frac{\partial g}{\partial \xi}+\frac{2\xi}{h_e}f'g\frac{\partial h_e}{\partial \xi}-2\xi g'\frac{\partial f}{\partial \xi}-fg'=\left(\frac{C}{\Pr}g'\right)'-\frac{2\xi\rho_e u_e f'}{\rho h_e}\frac{du_e}{d\xi}+\frac{u_e^2 C}{h_e}\left(f''\right)^2$$

Finally rearranging the terms we get,

$$\boxed{\left(\frac{C}{\Pr}g'\right)'+fg'=2\xi\left[f'\frac{\partial g}{\partial \xi}+\frac{f'g}{h_e}\frac{\partial h_e}{\partial \xi}-g'\frac{\partial f}{\partial \xi}+\frac{\rho_e u_e f'}{\rho h_e}\frac{du_e}{d\xi}\right]-\frac{u_e^2 C}{h_e}\left(f''\right)^2} \tag{26.7}$$

Hence, the governing Eqs. (25.9), (25.10) and (26.7) form the transformed set of equations for compressible hypersonic boundary layer.

The boundary conditions to be considered for solution are,

$\eta=0$, $f=f'=0$ and $g=g_w$ for isothermal wall boundary condition

While,

$\eta=0$, $f=f'=0$ and $g'=0$ for adiabatic wall boundary condition

The boundary condition at the edge of the boundary layer is,

$\eta \to \infty$, $f' = 1$ and $g = 1$

We can solve the transformed boundary layer equations which are partial differential equations. The necessary outcome for this is the velocity and enthalpy variation in the boundary layer. Since pressure is assumed to have same variation in the boundary layer as that at the outer flow, we can always evaluate the other thermodynamic properties in the boundary layer. Moreover the main parameters which we can evaluate are skin friction coefficient and Stanton Number.

We know that the skin friction coefficient is defined as,

$$C_f = \frac{\tau_W}{\frac{1}{2}\rho_e u_e^2}$$

Where, the wall shear can be calculated as,

$$\tau_W = \left(\mu \frac{\partial u}{\partial y}\right)_w$$

Hence,

$$C_f = \frac{2}{\rho_e u_e^2}\mu_w \left(\frac{\partial u}{\partial y}\right)_w$$

Using the expression given by Eq. (24.9) we can write the equation for skin friction coefficient as,

$$C_f = \frac{2}{\rho_e u_e^2}\mu_w \frac{u_e \rho_w}{\sqrt{2\xi}}\left(\frac{\partial u}{\partial \eta}\right)_w$$

$$C_f = \frac{2}{\rho_e u_e^2}\mu_w \frac{u_e^2 \rho_w}{\sqrt{2\xi}} f''(\xi, 0)$$

$$C_f = \frac{2\mu_w \rho_w}{\rho_e \sqrt{2\xi}} f''(\xi, 0) \tag{26.8}$$

We can also calculate the Stanton number from the solution of boundary layer equations. Here Stanton number is defined as,

$$S_t = \frac{q_W}{\rho_e u_e (h_{aw} - h_w)}$$

Here h_{aw} is the adiabatic wall enthalpy, which is the enthalpy at the wall when there is no heat transfer from the fluid to the wall. As well, h_w is the wall static enthalpy corresponding to the temperature of the wall at isothermal wall temperature condition. Here, q_W is the heat flux to the wall from the fluid which can be evaluated as,

$$q_W = \left(k\frac{\partial T}{\partial y}\right)_w \text{ or } q_W = \left(\frac{k}{c_p}\frac{\partial h}{\partial y}\right)_w$$

Assumption made here is the constancy of specific heat or calorically perfect gas. Thus using this expression for heatflux and from Eq. (24.9) and (25.13) we can calculate the Stanton number as,

$$S_t = \frac{1}{\rho_e u_e (h_{aw} - h_w)}\left(\frac{k}{c_p}\frac{\partial h}{\partial y}\right)_w$$

$$S_t = \frac{1}{\rho_e u_e (h_{aw} - h_w)}\left(\frac{k}{c_p}\frac{u_e \rho h_e}{\sqrt{2\xi}}\frac{\partial h}{\partial \eta}\right)_w \qquad (26.9)$$

$$S_t = \frac{1}{\rho_e u_e (h_{aw} - h_w)}\frac{k}{c_p}\frac{u_e \rho h_e}{\sqrt{2\xi}} g'(\xi, 0)$$

Lecture-27: Hypersonic flow over flat plate

27.1 Hypersonic flow over flat plate.

The equations (25.9)(25.10) and (26.7) form the boundary layer equations. Let's consider these equations for solution of hypersonic flow over flat plate. As we had seen earlier, there are two prominent boundary conditions considered for this flow over flat plate as,

Isothermal wall, T_w=const

Or an adiabatic wall, $\left(\dfrac{\partial T}{\partial y}\right)_w = 0$

All the freestream variables of the variables at the edge of the boundary layer are now assumed be constant. Hence The variables u_e, h_e, ρ_e are of constant values and independent of ξ and η. Therefore the governing equations reduce to,

$$(Cf'')' + ff'' = 2\xi\left(f'\frac{\partial f'}{\partial \xi} - \frac{\partial f}{\partial \xi} f'' \right) \qquad (27.1)$$

$$\left(\frac{C}{\Pr} g'\right)' + fg' = 2\xi\left[f'\frac{\partial g}{\partial \xi} - g'\frac{\partial f}{\partial \xi} \right] - \frac{u_e^2 C}{h_e}\left(f''\right)^2 \qquad (27.2)$$

These equations are the partial differential equations. Let's assume that the functions f and g are functions of Eqs.(4.40) and (4.41) are still partial differential equations . Let us assume that f and g are the functions of η only. Hence f and g are independent of ξ. For these assumptions the same equations reduce to,

$$\boxed{(Cf'')' + ff'' = 0} \qquad (27.3)$$

$$\boxed{\left(\frac{C}{\Pr} g'\right)' + fg' + \frac{u_e^2 C}{h_e}\left(f''\right)^2 = 0} \qquad (27.4)$$

These equations are single independent variable equations, hence are the non-linear ordinary differential equations. These equations are valid for a compressible boundary layer over a flat plate with constant wall conditions. Here both the constants

$C = \rho\mu / \rho_e\mu_e$ and $\Pr = \mu c_p / \kappa$ are meant for the local values in the boundary layer. We can you shooting technique for solving these equations. Thus obtained velocity and thermal boundary layers can be used to obtain the wall heat flux and shear stress in turn the Stanton number and skin friction coefficient. Here the ratio of Stanton number and skin friction coefficient can be approximated as the function of Prandtl's number by Reynolds analogy.

27.2 Hypersonic flow around a stagnation point

Hypersonic flow around a blunt body forms a stagnation point which can also be evaluated using boundary layer equations. Consider hypersonic flow around a blunt body which marks a stagnation region, as sketched in Fig. 27.1. Consider the flow to be 2D flow for simplicity; hence the span of the cylinder is infinity. Let X be the direction of freestream flow and R be the radius of curvature at the surface.

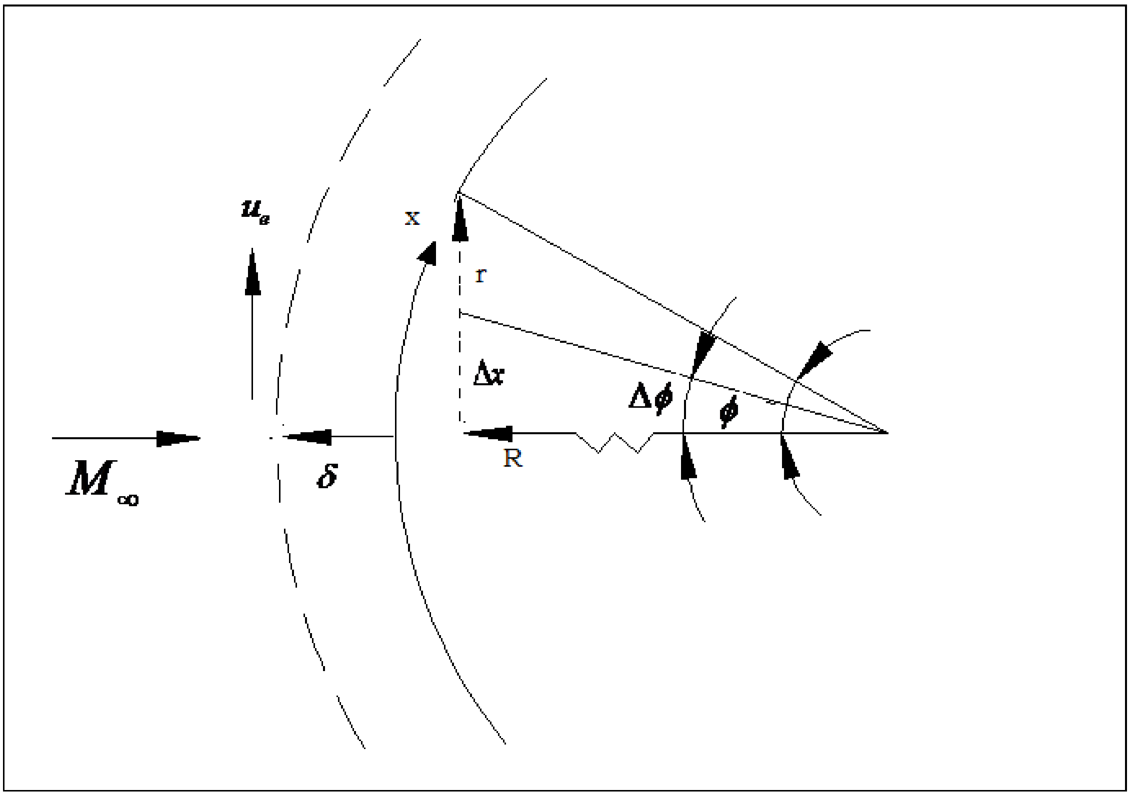

Fig. 27.1 Hypersonic flow around the stagnation point region [1].

Let us consider that f and g from Eq. (25.9) and (26.7) are functions of η alone,

Therefore,

$$\frac{\partial f'}{\partial \xi} = \frac{\partial f}{\partial \xi} = \frac{\partial g}{\partial \xi} = 0$$

This leads to the following changed in Eq. (25.9) and (26.7) as,

$$(Cf'')' + ff'' = \frac{2\xi}{u_e}[(f')^2 - \frac{\rho_e}{\rho}]\frac{du_e}{d\xi} \tag{27.5}$$

And

$$\left(\frac{C}{Pr}g'\right) + fg' = 2\xi[\frac{\rho_e u_e}{\rho h_e}f'\frac{du_e}{d\xi}] - C\frac{u_e}{h_e}(f'')^2 \tag{27.6}$$

These equations are still ξ dependent. Moreover, we can assume that the velocity at the edge of the boundary layer, u_e is very small and static enthalpy at the edge of the boundary layer is $h_e = h_0$ (stagnation enthalpy). These facts lead to the assumption that,

$$\frac{{u_e}^2}{h_e} \approx 0 \tag{27.7}$$

We can as well assume that, the flow velocity is the boundary layer at edge of the boundary layer at the stagnation point behind the normal shock is low as to be considered in the incompressible flow regime. Hence, we can use the result of incompressible and inviscid flow at the stagnation point, which expresses the boundary layer velocity as,

$$u_e = \left(\frac{du_e}{dx}\right)_s x \tag{27.8}$$

Here $\left(\frac{du_e}{dx}\right)_s$ is the velocity gradient at the stagnation point external to the boundary layer. Using these assumptions we can re-express the ξ as,

$$\xi = \int_0^x \rho_e u_e \mu_e dx = \int_0^x \rho_e \mu_e \left(\frac{du_e}{dx}\right)_s x dx$$

Or

$$\xi = \rho_e \mu_e \left(\frac{du_e}{dx}\right)_s \frac{x^2}{2} \tag{27.9}$$

The velocity gradient required for calculation here can be evaluated as,

$$\frac{du_e}{d\xi} = \left(\frac{du_e}{dx}\right)\left(\frac{dx}{d\xi}\right) = \frac{(du_e/dx)}{(d\xi/dx)} \tag{27.10}$$

But from the definition of ξ, Eq. (24.1) we know that,

$$\frac{d\xi}{dx} = \rho_e u_e \mu_e \tag{27.11}$$

Substituting this Eq. (27.11) into (27.10), we get

$$\frac{du_e}{d\xi} = \frac{1}{\rho_e u_e \mu_e}\frac{du_e}{dx} \tag{27.12}$$

From Eq. (27.11) into (27.12), we write,

$$\left(\frac{du_e}{dx}\right)_s = \frac{1}{\rho_e \mu_e x}$$

Or,

$$\left(\frac{du_e}{dx}\right)_s = \frac{1}{\mu_e \rho_e (du_e/dx)_s x}\left(\frac{du_e}{dx}\right)_s$$

Let's consider the term $(2\xi/u_e)\, du_e/d\xi$ which appears in the Eq. 27.1. We can re-write this term and can derive for the same as,

$$\frac{2\xi}{u_e}\frac{du_e}{d\xi} = \frac{2[\rho_e \mu_e (du_e/dx)_s (x^2/2)]}{(du_e/dx)_s x}\frac{1}{\rho_e \mu_e x} = 1$$

Similarly consider the term $(2\xi/u_e)(\rho_e u_e/\rho h_e)\, du_e/d\xi$ appearing in Eq. (27.2). We can re-arrange this term as well,

$$2\xi\frac{\rho_e u_e}{\rho h_e}\frac{du_e}{d\xi} = 2\frac{\rho_e}{\rho h_e}\left[\rho_e u_e \left(\frac{du_e}{dx}\right)_s \frac{x^2}{2}\right]\left[\left(\frac{du_e}{dx}\right)_s x\right]\left[\frac{1}{\rho_e \mu_e x}\right] = \frac{\rho_e}{\rho h_e}\left(\frac{du_e}{dx}\right)_s^2 x^2$$

However at the stagnation point, x=0. This fact leads to,

$$2\xi\frac{\rho_e u_e}{\rho h_e}\frac{du_e}{d\xi} = 0$$

Moreover, for a calorically perfect gas, we have,

$$\frac{\rho_e}{\rho} = \frac{p_e}{p}\frac{T_e}{T} = \frac{p_e}{p}\frac{h}{h_e} = \frac{h}{h_e} \equiv g$$

Using all these short expression for various terms involved in Eq.(27.1) and (27.2) we can get these equations as,

$$(Cf'')' + ff'' = (f')^2 - g \qquad (27.13)$$

$$\left(\frac{C}{Pr}g'\right)' + fg' = 0 \qquad (27.14)$$

These equations are special equation for stagnation point flow. These equations are independent of ξ. We can use numerical techniques shooting technique to solve these equations.

Lecture-28: Stagnation point flow field

28.1 Hypersonic flow around a stagnation point.

The known correlation for stagnation point heat flux for cylinder is

Cylinder: $$qe = 0.57\, Pr^{-0.6}(\rho_e\, \mu_e)^{1/2}\sqrt{\frac{du_e}{dx}}(h_{aw} - h_w) \tag{28.1}$$

This equation is valid for 2D configurations. Moreover for, sphere or axi-symmetric configuration the stagnation point heat flux can be predicted using,

Sphere: $$0.763 Pr^{-0.6}(\rho_e \mu_e)^{1/2}\sqrt{\frac{du_e}{dx}}(h_{aw} - h_w) \tag{28.2}$$

These equations are called as Fay and Riddle equations for stagnation point heat flux prediction. The stagnation point heat flux is more for sphere in comparison for that of the cylinder of same diameter. The main reason for this discrimination is the dimensionality of the flow. The hypersonic flow is two dimensional for flow over cylinder hence it has two possible direction for passing over the cylinder however the flow over the sphere is three dimensional. Due to the extra available dimension, flow passes around the object easily hence the shock stand off distance and the boundary layer thickness decrease for the sphere in comparison with the cylinder of same diameter. The decreased boundary layer thickness increases the gradient and hence the shear stress and heat flux at the wall for sphere in comparison with the cylinder.

The closer observation to the Eq. (28.1) and (28.2) suggests that the wall heat flux is propotional to the square root of the stream wise velocity gradient,($\frac{du_e}{dx}$) along the stagnation streamline.

$$dp_e = -\rho_e u_e du_e$$

Hence,

$$\frac{du_e}{dx} = -\frac{1}{\rho_e u_e}\frac{dp_e}{dx} \tag{28.3}$$

We can evaluate the pressure gradient of this equation from the known pressure variation given by Newtonian method.

$$C_p = 2\sin^2\theta$$

Here θ is flow deflection angle, the angle between a tangent at any point on the surface and the freestream direction. Let's define $\emptyset$ as the angle between freestream velocity and the local surface normal. Hence the pressure distribution gets transformed to,

$$C_p = 2\cos^2\varphi$$

From definition of pressure coefficient we have,

$$\frac{p_e - p_\infty}{q_\infty} = 2cos^2\emptyset$$

Or,

$$p_e = 2q_\infty\cos^2\varphi + p_\infty$$

Differentiating this equation w.r.t X we get,

$$\frac{dp_e}{dx} = -\frac{4q_\infty}{\rho_e u_e}\cos\varphi\sin\varphi\frac{d\varphi}{dx} \qquad (28.4)$$

Substituting the Eq. (28.4) in to the Eq. (28.3) we get,

$$\frac{du_e}{dx} = \frac{4q_\infty}{u_e\rho_e}cos\emptyset\ sin\emptyset\frac{d\emptyset}{dx} \qquad (28.5)$$

All the terms involved in this equation can be evaluated using various approximations.

$$u_e = \left(\frac{du_e}{dx}\right)_s \Delta x \qquad (28.6)$$

$cos\emptyset \approx 1$

$sin\emptyset \approx \emptyset \approx \Delta\emptyset \approx \frac{\Delta x}{R}$

$\frac{d\emptyset}{dx} = \frac{1}{R}$ (28.7)

$q_{\infty} = \frac{1}{2}(p_e - p_{\infty})$ (28.8)

Now, substituting Eqs. (28.6) – (28.8) in (28.5), we get,

$$\left(\frac{du_e}{dx}\right)^2 = \frac{2(p_e - p_{\infty})}{\rho_e \Delta x}\left(\frac{\Delta x}{R}\right)\left(\frac{1}{R}\right)$$

Or

$\frac{du_e}{dx} = \frac{1}{R}\sqrt{\frac{2(p_e - p_{\infty})}{\rho_{\infty}}}$ (28.9)

This is the approximate expression for velocity gradient encountered in the Eq. (28.1) or (28.2). The expression clearly suggests that the wall heat flux is inversely proportional to the nose radius or radius at the stagnation point. This is the main reason for having hypersonic vehicles being blunt nosed to reduce the heat load at the compromise of the drag.