Assignment 8 (Solutions)
NPTEL MOOC (Bayesian/ MMSE Estimation for MIMO/OFDM Wireless Communications)

1. Given data: $N = 4$, $L = 2$, $\sigma_h^2 = 1/4$, $\sigma^2 = 1/2$, mimimum cyclic prefix(CP) = L-1 = 1.
   The pilot symbols transmitted on all the subcarriers are given as $X(0) = 3 + 3j$, $X(1) = -2 - 2j$, $X(2) = -1 + j$, $X(3) = 2 - j$ and the received samples in the time domain be $y(0) = -1 - j$, $y(1) = 2 + 2j$, $y(2) = 3 - 2j$, $y(3) = 3 - 2j$. Time domain sample is generated by $N = 4$ pt IDFT and the kth sample is given by,

$$x(k) = \frac{1}{N} \sum_{l=0}^{N-1} X(l) e^{j2\pi \frac{kl}{N}}.$$

   The time domain samples $x(0)$, $x(1)$, $x(2)$ and $x(3)$ can be calculated as,

$$
\begin{aligned}
x(0) &= \frac{1}{4}\sum_{l=0}^{3} X(l)e^{j\frac{\pi}{2}0} \\
&= \frac{1}{4}\sum_{l=0}^{3} X(l) \\
&= \frac{1}{2} + \frac{1}{4}j, \\
x(1) &= \frac{1}{4}\sum_{l=0}^{3} X(l)e^{j\frac{\pi}{2}l} \\
&= \frac{1}{4}\{3+3j+j(-2-2j)-(-1+j)-j(2-j)\} \\
&= \frac{5}{4} - \frac{1}{2}j, \\
x(2) &= \frac{1}{4}\sum_{l=0}^{3} X(l)e^{j\pi l} \\
&= \frac{1}{4}\{3+3j-(-2-2j)+(-1+j)-(2-j)\} \\
&= \frac{1}{2} + \frac{7}{4}j, \\
x(3) &= \frac{1}{4}\sum_{l=0}^{3} X(l)e^{j\frac{3\pi}{2}l} \\
&= \frac{1}{4}\{3+3j-j(-2-2j)-(-1+j)+j(2-j)\} \\
&= \frac{3}{4} + \frac{3}{2}j.
\end{aligned}
$$

The transmitted block of samples in the time domain is given by,

$$x(3), x(0), x(1), x(2), x(3)$$

$$\frac{3}{4} + \frac{3}{2}j,\ \frac{1}{2} + \frac{1}{4}j,\ \frac{5}{4} - \frac{1}{2}j,\ \frac{1}{2} + \frac{7}{4}j,\ \frac{3}{4} + \frac{3}{2}j.$$

Ans (c)

2. The received symbols across the subcarriers are generated by $N = 4$ pt FFT and the kth sample is given by,

$$Y(l) = \sum_{k=0}^{N-1} y(k)e^{-j2\pi\frac{kl}{N}}.$$

So, the received samples $Y(0)$, $Y(1)$, $Y(2)$ and $Y(3)$ can be calculated as,

$$\begin{aligned}
Y(0) &= \sum_{k=0}^{3} y(k) e^{-j\frac{\pi}{2}0} \\
&= \sum_{k=0}^{3} y(k) \\
&= 7 - 3j, \\
Y(1) &= \sum_{k=0}^{3} y(k) e^{-j\frac{\pi}{2}k} \\
&= -1 - j - j(2+2j) - (3-2j) + j(3-2j) \\
&= 2j, \\
Y(2) &= \sum_{k=0}^{3} y(k) e^{-j\pi k} \\
&= -1 - j - (2+2j) + 3 - 2j - (3-2j) \\
&= -3 - 3j, \\
Y(3) &= \sum_{k=0}^{3} y(k) e^{-j\frac{\pi}{2}3k} \\
&= -1 - j + j(2+2j) - (3-2j) - j(3-2j) \\
&= -8.
\end{aligned}$$

The received symbols across the subcarriers are,

$$7 - 3j \ , 2j \ , -3 - 3j \ , -8.$$

Ans (b)

3. The system model is given by,

$$Y(l) = H(l)X(l) + V(l).$$

The LMMSE estimate of $H(l)$ is given by,

$$\hat{H}(l) = \frac{L\sigma_h^2 X^*(l)}{L\sigma_h^2 |X(l)|^2 + N\sigma^2} Y(l).$$

The LMMSE estimates $\hat{H}(0)$, $\hat{H}(1)$ of the channel coefficients $H(0)$, $H(1)$ across subcarriers 0, 1 respectively can be calculated as,

$$\begin{aligned}\hat{H}(0) &= \frac{(3-3j)(7-3j)}{22} \\ &= \frac{6}{11} - \frac{15}{11}j, \\ \hat{H}(1) &= \frac{(-2+2j)(2j)}{12} \\ &= -\frac{1}{3} - \frac{1}{3}j.\end{aligned}$$

Ans (d)

4. The system model is given by,

$$Y(l) = H(l)X(l) + V(l).$$

The LMMSE estimate of $H(l)$ is given by,

$$\hat{H}(l) = \frac{L\sigma_h^2 X^*(l)}{L\sigma_h^2 |X(l)|^2 + N\sigma^2} Y(l).$$

The LMMSE estimates $\hat{H}(2)$, $\hat{H}(3)$ of the channel coefficients $H(2)$, $H(3)$ across subcarriers 2, 3 respectively can be calculated as,

$$\begin{aligned}\hat{H}(2) &= \frac{(-1-j)(-3-3j)}{6} \\ &= j, \\ \hat{H}(3) &= \frac{(2+j)(-8)}{9} \\ &= -\frac{16}{9} - \frac{8}{9}j.\end{aligned}$$

Ans (a)

5. The LMMSE estimate of $H(l)$ is given by,

$$\hat{H}(l) = \frac{L\sigma_h^2 X^*(l)}{L\sigma_h^2 |X(l)|^2 + N\sigma^2} Y(l).$$

At high SNR, i.e $L\sigma_h^2|X(l)|^2 >> N\sigma^2$. The ML estimate of $H(l)$ is given by,

$$\hat{H}(l) = \frac{1}{X(l)} Y(l).$$

The The ML estimates $\hat{H}(2)$, $\hat{H}(3)$ of the channel coefficients $H(2)$, $H(3)$ across subcarriers 2, 3 respectively can be calculated as,

$$\begin{aligned}\hat{H}(2) &= \frac{-3-3j}{-1+j} \\ &= 3j, \\ \hat{H}(3) &= \frac{-8}{2-j} \\ &= -\frac{16}{5} - \frac{8}{5}j.\end{aligned}$$

Ans (b)

6. The MSE of the LMMSE estimate is given by,

$$\mathrm{E}\{|\hat{H}(l) - H(l)|^2\} = \frac{1}{\frac{1}{N\sigma^2/|X(l)|^2} + \frac{1}{L\sigma_h^2}}.$$

The MSEs of the LMMSE estimates $\hat{H}(0)$, $\hat{H}(1)$, $\hat{H}(2)$, $\hat{H}(3)$ of the channel coefficients $H(0)$, $H(1)$, $H(2)$, $H(3)$ across subcarriers 0, 1, 2 , 3 respectively can be calculated as

$$\begin{aligned}\mathrm{E}\{|\hat{H}(0) - H(0)|^2\} &= \frac{1}{\frac{1}{4\sigma^2/|X(0)|^2} + \frac{1}{2\sigma_h^2}} \\ &= \frac{1}{11}, \\ \mathrm{E}\{|\hat{H}(1) - H(1)|^2\} &= \frac{1}{\frac{1}{4\sigma^2/|X(1)|^2} + \frac{1}{2\sigma_h^2}} \\ &= \frac{1}{6}, \\ \mathrm{E}\{|\hat{H}(2) - H(2)|^2\} &= \frac{1}{\frac{1}{4\sigma^2/|X(2)|^2} + \frac{1}{2\sigma_h^2}} \\ &= \frac{1}{3}, \\ \mathrm{E}\{|\hat{H}(3) - H(3)|^2\} &= \frac{1}{\frac{1}{4\sigma^2/|X(3)|^2} + \frac{1}{2\sigma_h^2}} \\ &= \frac{2}{9}.\end{aligned}$$

Ans (a)

7. The noise sample $V(l)$ on the lth subcarrier is given by,

$$V(l) = \sum_{k=0}^{N-1} v(k) e^{-j2\pi \frac{kl}{N}},$$

where mean of $V(l)$ is given by,

$$\begin{aligned}
\mathrm{E}\{V(l)\} &= \mathrm{E}\Big\{ \sum_{k=0}^{N-1} v(k) e^{-j2\pi \frac{kl}{N}} \Big\} \\
&= \sum_{k=0}^{N-1} \mathrm{E}\{v(k)\} e^{-j2\pi \frac{kl}{N}} \\
&= 0,
\end{aligned}$$

and variance of $V(l)$ is given by,

$$\begin{aligned}
\mathrm{E}\{|V(l)|^2\} &= \mathrm{E}\{V(l)V^*(l)\} \\
&= \mathrm{E}\Bigg\{ \Bigg(\sum_{k=0}^{N-1} v(k) e^{-j2\pi \frac{kl}{N}} \Bigg) \Bigg(\sum_{\tilde{k}=0}^{N-1} v(\tilde{k}) e^{-j2\pi \frac{\tilde{k}l}{N}} \Bigg) \Bigg\} \\
&= N\sigma^2 \\
&= 2.
\end{aligned}$$

So, the noise sample $V(l)$ on the lth subcarrier is zero-mean Gaussian with variance 2.
Ans (d)

8. The channel coefficient $H(l)$ on the lth subcarrier is given by,

$$H(l) = \sum_{k=0}^{N-1} h(k) e^{-j2\pi \frac{kl}{N}},$$

where mean of $V(l)$ is given by,

$$\begin{aligned}
\mathrm{E}\{H(l)\} &= \mathrm{E}\Big\{ \sum_{k=0}^{L-1} h(k) e^{-j2\pi \frac{kl}{N}} \Big\} \\
&= \sum_{k=0}^{L-1} \mathrm{E}\{h(k)\} e^{-j2\pi \frac{kl}{N}} \\
&= 0,
\end{aligned}$$

and variance of $V(l)$ is given by,

$$\begin{aligned}
\mathrm{E}\{|H(l)|^2\} &= \mathrm{E}\{H(l)H^*(l)\} \\
&= \mathrm{E}\left\{\left(\sum_{k=0}^{L-1} h(k)e^{-j2\pi\frac{kl}{N}}\right)\left(\sum_{\tilde{k}=0}^{L-1} h(\tilde{k})e^{-j2\pi\frac{\tilde{k}l}{N}}\right)\right\} \\
&= L\sigma_h^2 \\
&= \frac{1}{2}.
\end{aligned}$$

So, the channel coefficient $H(l)$ on the lth subcarrier is zero-mean with variance $\frac{1}{2}$.
Ans (b)

9. The LMMSE estimates of the channel tap $\hat{h}(k)$ is given by,

$$\hat{h}(k) = \frac{1}{N}\sum_{l=0}^{N-1} \hat{H}(l)e^{j2\pi\frac{kl}{N}}.$$

The LMMSE estimates $\hat{H}(0)$, $\hat{H}(1)$, $\hat{H}(2)$, $\hat{H}(3)$ of the channel coefficients $H(0)$, $H(1)$, $H(2)$, $H(3)$ across subcarriers 0, 1, 2 and 3 respectively are,

$$\begin{aligned}
\hat{H}(0) &= \frac{(3-3j)(7-3j)}{22} \\
&= \frac{6}{11} - \frac{15}{11}j, \\
\hat{H}(1) &= \frac{(-2+2j)(2j)}{12} \\
&= -\frac{1}{3} - \frac{1}{3}j, \\
\hat{H}(2) &= \frac{(-1-j)(-3-3j)}{6} \\
&= j, \\
\hat{H}(3) &= \frac{(2+j)(-8)}{9} \\
&= -\frac{16}{9} - \frac{8}{9}j.
\end{aligned}$$

The LMMSE estimates of the channel taps $\hat{h}(0)$, $\hat{h}(1)$ respectively can be calculated as,

$$\begin{aligned}\hat{h}(0) &= \frac{1}{4}\sum_{l=0}^{3}\hat{H}(l)e^{j\frac{\pi}{2}0}\\ &= \frac{1}{4}\left\{\frac{6}{11} - \frac{15}{11}j - \frac{1}{3} - \frac{1}{3}j + j - \frac{16}{9} - \frac{8}{9}j\right\}\\ &= -\frac{155}{396} - \frac{157}{396}j,\\ \hat{h}(1) &= \frac{1}{4}\sum_{l=0}^{3}\hat{H}(l)e^{j\frac{\pi}{2}l}\\ &= \frac{1}{4}\left\{\frac{6}{11} - \frac{15}{11}j + j\left(-\frac{1}{3} - \frac{1}{3}j\right) - j - j\left(-\frac{16}{9} - \frac{8}{9}j\right)\right\}\\ &= -\frac{1}{396} - \frac{91}{396}j.\end{aligned}$$

Ans (c)

10. As seen from solution to 6 above, variance in estimate of $H(1)$ is $\frac{1}{6}$. As we know, since the pilot block is repeated M times, the variance corresponding to M pilots on each subcarrier is $\frac{\frac{1}{6}}{M} = \frac{1}{6M}$. Let $\hat{H}_R(1)$ denotes the real part of the estimate $\hat{H}(1)$. Further, $\hat{H}_R(1) - H_R(1)$ gives the estimation error in the real part of the estimate. Therefore, $H_R(1) - \hat{H}_R(1)$ is distributed as a zero-mean Gaussian with variance $\frac{1}{12M}$. Therefore, $\frac{\hat{H}_R(1)-H_R(1)}{\sqrt{\frac{1}{12M}}}$ is a zero-mean unit-variance Gaussian RV. Therefore, probability that the real part of the MMSE estimate $\hat{H}(1)$ lies within a radius 1/8 of the real part of the true parameter $H(1)$ can be calculated as follows,

$$\begin{aligned}&\Pr\left(|\hat{H}_R(1) - H_R(1)| \le \frac{1}{8}\right) = \Pr\left(\frac{|\hat{H}_R(1) - H_R(1)|}{\sqrt{\frac{1}{12M}}} \le \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}}\right)\\ &= 1 - \Pr\left(\frac{|\hat{H}_R(1) - H_R(1)|}{\sqrt{\frac{1}{12M}}} \ge \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}}\right)\\ &= 1 - \left\{\Pr\left(\frac{\hat{H}_R(1) - H_R(1)}{\sqrt{\frac{1}{12M}}} \ge \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}}\right) + \Pr\left(\frac{\hat{H}_R(1) - H_R(1)}{\sqrt{\frac{1}{12M}}} \le -\frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}}\right)\right\}\\ &= 1 - 2\Pr\left(\frac{\hat{H}_R(1) - H_R(1)}{\sqrt{\frac{1}{12M}}} \ge \frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}}\right) = 1 - 2Q\left(\frac{\frac{1}{8}}{\sqrt{\frac{1}{12M}}}\right) = 1 - 2Q\left(\sqrt{\frac{12M}{64}}\right)\end{aligned}$$

Further, since the errors in the real and imaginary parts are independent as they are Gaussian, the probability that both the real and imaginary parts of the MMSE estimate $\hat{H}(1)$ lie within a radius of 1/8 from the real and imaginary parts of the true parameter $H(1)$ respectively is $\left(1 - 2Q\left(\sqrt{\frac{12M}{64}}\right)\right)^2$.

Therefore, probability that both the real and imaginary parts of the MMSE estimate $\hat{H}(1)$ lie within a radius of 1/8 from the real and imaginary parts of the true parameter $H(1)$ is greater than 99.99% is

$$\begin{aligned}
\left(\Pr\left(|\hat{H}_R(1) - H_R(1)| \leq \frac{1}{8}\right)\right)^2 &\geq .9999 \\
1 - 2Q\left(\sqrt{\frac{12M}{64}}\right) &\geq .9999 \\
2Q\left(\sqrt{\frac{12M}{64}}\right) &\leq 10^{-4} \\
Q\left(\sqrt{\frac{12M}{64}}\right) &\leq 5 \times 10^{-5} \\
\sqrt{\frac{12M}{64}} &\geq Q^{-1}(5 \times 10^{-5}) \\
M &\geq \frac{16}{3}(Q^{-1}(5 \times 10^{-5}))^2
\end{aligned}$$

Ans (d)