Assignment 6 (Solutions)
NPTEL MOOC (Bayesian/ MMSE Estimation for MIMO/OFDM Wireless Communications)

1. The input output model for the kth instant can be written as,

$$\mathbf{y}^T(k) = \mathbf{x}^T(k)\mathbf{H} + \mathbf{v}^T(k).$$

By stacking all such N received vectors, the system model can be written as,

$$\mathbf{Y} = \mathbf{X}\mathbf{H} + \mathbf{V},$$

where covariance matrix of $\mathbf{H} = \mathrm{E}\{\mathbf{H}\mathbf{H}^T\} = \mathrm{R}\sigma_h^2\mathbf{I}_{\mathrm{MXM}}$. Similarly, the covariance matrix of $\mathbf{V} = \mathrm{E}\{\mathbf{V}\mathbf{V}^T\} = \mathrm{R}\sigma^2\mathbf{I}_{\mathrm{NXN}}$. The LMMSE estimate of the MIMO channel matrix $\mathbf{H}$ is given by,

$$\hat{\mathbf{H}} = \mathbf{R}_{\mathrm{HY}}\mathbf{R}_{\mathrm{YY}}^{-1}\mathbf{Y}.$$

The covariance matrix of $\mathbf{Y}$ can be calculated as,

$$\begin{aligned}
\mathbf{R}_{\mathrm{YY}} &= \mathrm{E}\{\mathbf{Y}\mathbf{Y}^T\} \\
&= \mathrm{E}\{(\mathbf{X}\mathbf{H} + \mathbf{V})(\mathbf{X}\mathbf{H} + \mathbf{V})^T\} \\
&= \mathrm{E}\{\mathbf{X}\mathbf{H}\mathbf{H}^T\mathbf{X}^T\} + \mathrm{E}\{\mathbf{V}\mathbf{H}^T\mathbf{X}^T\} + \mathrm{E}\{\mathbf{X}\mathbf{H}\mathbf{V}^T\} + \mathrm{E}\{\mathbf{V}\mathbf{V}^T\} \\
&= \mathrm{R}\sigma_h^2\mathbf{X}\mathbf{X}^T + \mathrm{R}\sigma^2\mathbf{I}.
\end{aligned}$$

Similarly, the covariance matrix of $\mathbf{R}_{\mathrm{HY}}$ can be calculated as,

$$\begin{aligned}
\mathbf{R}_{\mathrm{HY}} &= \mathrm{E}\{\mathbf{H}\mathbf{Y}^T\} \\
&= \mathrm{E}\{\mathbf{H}(\mathbf{X}\mathbf{H} + \mathbf{V})^T\} \\
&= \mathrm{R}\sigma_h^2\mathbf{X}^T.
\end{aligned}$$

So, $\hat{\mathbf{H}}$ can be written as,

$$\begin{aligned}\hat{\mathbf{H}} &= \mathrm{R}\sigma_h^2\mathbf{X}^T(\mathrm{R}\sigma_h^2\mathbf{X}\mathbf{X}^T + \mathrm{R}\sigma^2\mathbf{I})^{-1}\mathbf{Y}\\ &= \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\mathbf{X}^T\mathbf{Y}\\ &= \left(\frac{\mathbf{X}^T\mathbf{X}}{\mathrm{R}\sigma^2} + \frac{\mathbf{I}_M}{\mathrm{R}\sigma_h^2}\right)^{-1}\frac{\mathbf{X}^T\mathbf{Y}}{\mathrm{R}\sigma^2}\end{aligned}$$

Ans (a)

2. For system model

$$\mathbf{Y} = \mathbf{X}\mathbf{H} + \mathbf{V},$$

the LMMSE estimate is given by,

$$\hat{\mathbf{H}} = \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\mathbf{X}^T\mathbf{Y}.$$

By assuming $\sigma_h^2\mathbf{X}^T\mathbf{X} >> \sigma^2\mathbf{I}$, the ML estimate can be obtained as,

$$\begin{aligned}\hat{\mathbf{H}} &= \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}\\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}.\end{aligned}$$

Ans (b)

3. The error covariance of the LMMSE estimate of the MIMO channel matrix $\mathbf{H}$ is given by,

$$\mathrm{E}\{(\hat{\mathbf{H}} - \mathbf{H})(\hat{\mathbf{H}} - \mathbf{H})^T\} = \mathbf{R}_{\mathrm{HH}} - \mathbf{R}_{\mathrm{HY}}\mathbf{R}_{\mathrm{YY}}^{-1}\mathbf{R}_{\mathrm{HY}}.$$

The covariance matrices $\mathbf{R}_{\mathrm{HH}},\ \mathbf{R}_{\mathrm{HY}}$ and $\mathbf{R}_{\mathrm{YY}}$ are as follows,

$$\begin{aligned}\mathbf{R}_{\mathrm{HH}} &= \mathrm{R}\sigma_h^2\mathbf{I},\\ \mathbf{R}_{\mathrm{HY}} &= \mathrm{R}\sigma_h^2\mathbf{X}^T = \mathbf{R}_{\mathrm{YH}}^T,\\ \mathbf{R}_{\mathrm{YY}} &= \mathrm{R}\sigma_h^2\mathbf{X}\mathbf{X}^T + \mathrm{R}\sigma^2\mathbf{I}.\end{aligned}$$

After substituting the covariance matrices in the above expression, the error covariance of the LMMSE estimate can be calculated as

$$\begin{aligned}\mathrm{E}\{(\hat{\mathbf{H}} - \mathbf{H})(\hat{\mathbf{H}} - \mathbf{H})^T\} &= \mathrm{R}\sigma_h^2\mathbf{I} - \mathrm{R}\sigma_h^2\mathbf{X}^T(\mathrm{R}\sigma_h^2\mathbf{X}\mathbf{X}^T + \mathrm{R}\sigma^2\mathbf{I})^{-1}\mathbf{X}\mathrm{R}\sigma_h^2\\ &= \mathrm{R}\sigma_h^2\mathbf{I} - \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}\mathrm{R}\sigma_h^2\\ &= \mathrm{R}\sigma_h^2\mathbf{I} - \sigma_h^2\mathrm{R}(\mathbf{I} - \sigma^2\mathbf{I}(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1})\\ &= \mathrm{R}\sigma_h^2\sigma^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\\ &= \left(\frac{\mathbf{X}^T\mathbf{X}}{\mathrm{R}\sigma^2} + \frac{\mathbf{I}}{\mathrm{R}\sigma_h^2}\right)^{-1}.\end{aligned}$$

Ans (d)

4. The error covariance of the LMMSE estimate is given by,

$$\mathrm{E}\{(\hat{\mathbf{H}} - \mathbf{H})(\hat{\mathbf{H}} - \mathbf{H})^T\} = \left(\frac{\mathbf{X}^T\mathbf{X}}{\mathrm{R}\sigma^2} + \frac{\mathbf{I}}{\mathrm{R}\sigma_h^2} \right)^{-1}.$$

By assuming $\sigma_h^2 \mathbf{X}^T\mathbf{X} >> \sigma^2 \mathbf{I}$, the covariance of the ML estimate can be obtained as,

$$\mathrm{E}\{(\hat{\mathbf{H}} - \mathbf{H})(\hat{\mathbf{H}} - \mathbf{H})^T\} = \left(\frac{\mathbf{X}^T\mathbf{X}}{\mathrm{R}\sigma^2} \right)^{-1}.$$

Ans (c)

5. The channel coefficients are IID Gaussian with dB variance $\sigma_h^2 = 0$ dB $= 1$ and the noise samples are also IID Gaussian with dB noise variance $\sigma^2 = -3$ dB $= \frac{1}{2}$. The pilot matrix $\mathbf{X}$ and the observation matrix $\mathbf{Y}$ are given by,

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & 2 & -1 \end{bmatrix},$$

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix},$$

$$(\sigma_h^2 \mathbf{X}^T\mathbf{X} + \sigma^2 \mathbf{I})^{-1} = \begin{bmatrix} \frac{2}{15} & 0 \\ 0 & \frac{2}{15} \end{bmatrix},$$

$$\mathbf{X}^T\mathbf{Y} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -3 \\ -6 & 2 & -2 \end{bmatrix}.$$

The LMMSE estimate of the MIMO channel matrix $\mathbf{H}$ is given by,

$$\hat{\mathbf{H}} = \sigma_h^2 (\sigma_h^2 \mathbf{X}^T\mathbf{X} + \sigma^2 \mathbf{I})^{-1} \mathbf{X}^T\mathbf{Y}$$

$$= \begin{bmatrix} 0 & 2/15 & -2/5 \\ -4/5 & 4/15 & -4/15 \end{bmatrix}.$$

Ans (a)

6. The ML estimate of the MIMO channel matrix $\mathbf{H}$ is given by,

$$\hat{\mathbf{H}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$
$$= \begin{bmatrix} 0 & 1/7 & -3/7 \\ -6/7 & 2/7 & -2/7 \end{bmatrix}.$$

Ans (b)

7. The error covariance of the LMMSE estimate of the MIMO channel matrix $\mathbf{H}$ is given by,

$$\mathrm{E}\{(\hat{\mathbf{H}} - \mathbf{H})(\hat{\mathbf{H}} - \mathbf{H})^T\} = \left(\frac{\mathbf{X}^T\mathbf{X}}{\mathrm{R}\sigma^2} + \frac{\mathbf{I}}{\mathrm{R}\sigma_h^2}\right)^{-1}.$$

After substituting all the values, we obtain

$$\mathrm{E}\{(\hat{\mathbf{H}} - \mathbf{H})(\hat{\mathbf{H}} - \mathbf{H})^T\} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix}.$$

Ans (c)

8. Channel equalization refers to removing the effect of ISI.
Ans (d)

9. For an Inter Symbol Interference channel, received symbol at kth instant is given by,

$$y(k) = \frac{3}{2}\, x(k) - \frac{1}{2}\, x(k-1) + v(k).$$

Comparing the above equation with L=2 tap wireless channel given below,

$$y(k) = h(0)x(k) + h(1)x(k-1) + v(k),$$

we obtain, $h(0) = \frac{3}{2}$ and $h(1) = -\frac{1}{2}$. Considering a $r = 3$ tap equalizer based on $y(k),\ y(k+1)$ and $y(k+2)$, model can be formulated as,

$$\begin{bmatrix} y(k+2) \\ y(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} h(0) & h(1) & 0 & 0 \\ 0 & h(0) & h(1) & 0 \\ 0 & 0 & h(0) & h(1) \end{bmatrix} \begin{bmatrix} x(k+2) \\ x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix} + \begin{bmatrix} v(k+2) \\ v(k+1) \\ v(k) \end{bmatrix}$$

$$\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k) + \mathbf{v}(k).$$

So, the effective channel matrix $\mathbf{H}$ for this scenario can be written as,

$$\mathbf{H} = \begin{bmatrix} 3/2 & -1/2 & 0 & 0 \\ 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & 3/2 & -1/2 \end{bmatrix}.$$

Ans (c)

10. The system model can be written as,

$$\mathbf{y} = \mathbf{X}\mathbf{h} + \mathbf{v},$$

where the channel vector $\mathbf{h}$ has mean and covariance matrix as $\mathrm{E}\{\mathbf{h}\} = \mathbf{0}$ and $\mathrm{E}\{\mathbf{h}\mathbf{h}^T\} = \mathbf{R}_h$, respectively. Similarly, noise vector $\mathbf{v}$ has $\mathrm{E}\{\mathbf{v}\} = \mathbf{0}$ and $\mathrm{E}\{\mathbf{v}\mathbf{v}^T\} = \mathbf{R}_v$. Also, $\mu_y = \mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{\mathbf{X}\mathbf{h} + \mathbf{v}\} = \mathbf{0}$. The observation covariance matrix is given by,

$$\begin{aligned}
\mathbf{R}_{yy} &= \mathrm{E}\{\mathbf{y}\mathbf{y}^T\} \\
&= \mathrm{E}\{(\mathbf{X}\mathbf{h} + \mathbf{v})(\mathbf{X}\mathbf{h} + \mathbf{v})^T\} \\
&= \mathbf{X}\mathrm{E}\{\mathbf{h}\mathbf{h}^T\}\mathbf{X}^T + \mathrm{E}\{\mathbf{v}\mathbf{h}^T\}\mathbf{X}^T + \mathbf{X}\mathrm{E}\{\mathbf{h}\mathbf{v}^T\} + \mathrm{E}\{\mathbf{v}\mathbf{v}^T\} \\
&= \mathbf{X}\mathbf{R}_h\mathbf{X}^T + \mathbf{R}_v.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{R}_{hy} &= \mathrm{E}\{\mathbf{h}\mathbf{y}^T\} \\
&= \mathrm{E}\{\mathbf{h}(\mathbf{X}\mathbf{h} + \mathbf{v})^T\} \\
&= \mathrm{E}\{\mathbf{h}\mathbf{h}^T\}\mathbf{X}^T + \mathrm{E}\{\mathbf{h}\mathbf{v}^T\} \\
&= \mathbf{R}_h\mathbf{X}^T = (\mathbf{R}_{yh})^T.
\end{aligned}$$

The LMMSE estimate of parameter $\mathbf{h}$ is given by

$$\hat{\mathbf{h}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{y}.$$

Substituting the values of the covariance matrices in the above expression, we obtain

$$\hat{\mathbf{h}} = \mathbf{R}_h\mathbf{X}^T(\mathbf{X}\mathbf{R}_h\mathbf{X}^T + \mathbf{R}_v)^{-1}\mathbf{y}. \tag{1}$$

Using the Woodbury matrix identity, we get

$$\begin{aligned}
&\mathbf{R}_h\mathbf{X}^T(\mathbf{X}\mathbf{R}_h\mathbf{X}^T + \mathbf{R}_v)^{-1} = \mathbf{R}_h\mathbf{X}^T(\mathbf{R}_v^{-1} - \mathbf{R}_v^{-1}\mathbf{X}(\mathbf{R}_h^{-1} + \mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{R}_v^{-1}) \\
&= \mathbf{R}_h\mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{y} - \mathbf{R}_h\mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X}(\mathbf{R}_h^{-1} + \mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{R}_v^{-1} \\
&= \mathbf{R}_h\mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{y} - \mathbf{R}_h(\mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X} + \mathbf{R}_h^{-1} - \mathbf{R}_h^{-1})(\mathbf{R}_h^{-1} + \mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{R}_v^{-1} \\
&= (\mathbf{R}_h^{-1} + \mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{R}_v^{-1}
\end{aligned}$$

After substituting the above expression in the equation (1), we obtain

$$\hat{\mathbf{h}} = (\mathbf{R}_h^{-1} + \mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{y}$$

The error covariance for the LMMSE estimate of $\mathbf{h}$ is,

$$\mathrm{E}\{(\hat{\mathbf{h}} - \mathbf{h})(\hat{\mathbf{h}} - \mathbf{h})^T\} = \mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}.$$

From the solutions to the above problem $\mathbf{R}_{hh}$, $\mathbf{R}_{hy}$ and $\mathbf{R}_{yy}$ can be written as,

$$\begin{aligned}
\mathbf{R}_{hh} &= \mathbf{R}_h, \\
\mathbf{R}_{hy} &= \mathbf{R}_h\mathbf{X}^T = (\mathbf{R}_{yh})^T, \\
\mathbf{R}_{yy} &= \mathbf{X}\mathbf{R}_h\mathbf{X}^T + \mathbf{R}_v.
\end{aligned}$$

Substituting the values in the above equation, we obtain

$$\begin{aligned}
\mathrm{E}\{(\hat{\mathbf{h}} - \mathbf{h})(\hat{\mathbf{h}} - \mathbf{h})^T\} &= \mathbf{R}_h - \mathbf{R}_h\mathbf{X}^T(\mathbf{X}\mathbf{R}_h\mathbf{X}^T + \mathbf{R}_v)^{-1}\mathbf{X}\mathbf{R}_h \\
&= (\mathbf{R}_h^{-1} + \mathbf{X}^T\mathbf{R}_v^{-1}\mathbf{X})^{-1}.
\end{aligned}$$

Ans (a)