1. Linear estimate of channel vector $\mathbf{h}$ is, linear combination of observations, i.e.,

$$\hat{\mathbf{h}} = c_1 y(1) + c_2 y(2) + ... + c_N y(N) = \mathbf{c}^T \mathbf{y}$$

, where, $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_N \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y(1) \\ y(2) \\ . \\ . \\ y(N) \end{bmatrix}$

Mean Squared Error,

$$\text{MSE} = E\left\{ \left\| \hat{\mathbf{h}} - \mathbf{h} \right\|^2 \right\}$$

Minimum Mean Squared Error,

$$\text{MMSE} = min_{\hat{h}} E\left\{ \left\| \hat{\mathbf{h}} - \mathbf{h} \right\|^2 \right\}$$

For LMMSE, $\hat{\mathbf{h}} = \mathbf{c}^T \mathbf{y}$.
Hence, the minimization problem for LMMSE estimate is,

$$min_{\hat{h}} E\left\{ \left\| \hat{\mathbf{h}} - \mathbf{h} \right\|^2 \right\},$$

where $\hat{\mathbf{h}} = \mathbf{c}^T \mathbf{y}$.

Ans(c)

2. Considering multi-antenna channel problem,

$$\mathbf{y} = \mathbf{X}\mathbf{h} + \mathbf{v}$$

where, $\mathbf{y}$ is Nx1 observation vector, $\mathbf{X}$ is NxM pilot matrix, $\mathbf{h}$ is Mx1 channel vector, $\mathbf{v}$ is Nx1 noise vector, and N= number of pilot vectors, M= number of antennas. Also, given that $E\{h_i\} = 0$, $E\{|h_i|^2\} = \sigma_h^2$, i.e.,

$$E\{\mathbf{h}\mathbf{h}^T\} = \sigma_h^2 \mathbf{I}$$

Also,

$$\mathbf{R}_{yy} = E\{\mathbf{y}\mathbf{y}^T\} = \sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}_{MXM},$$
$$\mathbf{R}_{hy} = E\{\mathbf{h}\mathbf{y}^T\} = \sigma_h^2 \mathbf{X}^T$$

LMMSE estimate of channel vector $\mathbf{h}$ is,

$$\hat{h} = \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \mathbf{y}$$

Substituting the values of $\mathbf{R}_{hy}$ and $\mathbf{R}_{yy}$,and Using the identity,

$$\sigma_h^2 \mathbf{X}^T (\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}_{MXM})^{-1} = \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I}_{MXM})^{-1} \mathbf{X}^T$$

we obtain,

$$\begin{aligned} \hat{h} &= \sigma_h^2 \mathbf{X}^T (\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}_{MXM})^{-1} \mathbf{y} \\ &= \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I}_{MXM})^{-1} \mathbf{X}^T \mathbf{y} \end{aligned}$$

Ans(a)

3. LMMSE estimate,

$$\hat{h} = \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\mathbf{X}^T\mathbf{y}$$

For high Signal to Noise Ratio(SNR) scenario, $\sigma_h^2\mathbf{X}^T\mathbf{X} >> \sigma^2\mathbf{I}$, thus, LMMSE estimate reduces to,

$$\begin{aligned}\hat{h} &= \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}\\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}\end{aligned}$$

For low Signal to Noise Ratio(SNR) scenario, $\sigma_h^2\mathbf{X}^T\mathbf{X} << \sigma^2\mathbf{I}$, thus, LMMSE estimate reduces to,

$$\begin{aligned}\hat{h} &= \sigma_h^2(\sigma^2\mathbf{I})^{-1}\mathbf{X}^T\mathbf{y}\\ &= \frac{\sigma_h^2}{\sigma^2}\mathbf{X}^T\mathbf{y}\end{aligned}$$

Ans(b)

4. The error covariance for the LMMSE estimate of $\mathbf{h}$ is,

$$\mathrm{E}\{(\hat{\mathbf{h}} - \mathbf{h})(\hat{\mathbf{h}} - \mathbf{h})^T\} = \mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}.$$

From the solutions to the above problem $\mathbf{R}_{hh}$, $\mathbf{R}_{hy}$ and $\mathbf{R}_{yy}$ can be written as,

$$\begin{aligned}\mathbf{R}_{hh} &= \sigma_h^2\mathbf{I},\\ \mathbf{R}_{hy} &= \sigma_h^2\mathbf{X}^T = (\mathbf{R}_{yh})^T,\\ \mathbf{R}_{yy} &= \sigma_h^2\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{I}.\end{aligned}$$

Substituting the values in the above equation, we obtain

$$\begin{aligned}\mathrm{E}\{(\hat{\mathbf{h}} - \mathbf{h})(\hat{\mathbf{h}} - \mathbf{h})^T\} &= \sigma_h^2\mathbf{I} - \sigma_h^2\mathbf{X}^T(\sigma_h^2\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{I})^{-1}\sigma_h^2\mathbf{X}\\ &= \sigma_h^2\mathbf{I} - \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\mathbf{X}^T\sigma_h^2\mathbf{X}\\ &= \sigma_h^2\mathbf{I} - \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I} - \sigma^2\mathbf{I})\\ &= \sigma^2\sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\\ &= \left(\frac{\mathbf{X}^T\mathbf{X}}{\sigma^2} + \frac{\mathbf{I}}{\sigma_h^2}\right)^{-1}.\end{aligned}$$

Ans (c)

5. For a multi-antenna channel estimation scenario with $N = 4$ pilot vectors

$$\mathbf{x}(1) = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{x}(2) = \begin{bmatrix}1\\-2\end{bmatrix}, \mathbf{x}(3) = \begin{bmatrix}2\\1\end{bmatrix}, \mathbf{x}(4) = \begin{bmatrix}1\\-1\end{bmatrix}.$$

The corresponding pilot matrix $\mathbf{X}$ is given by,

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}(1)^T \\ \mathbf{x}(2)^T \\ \mathbf{x}(3)^T \\ \mathbf{x}(4)^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix},$$

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1} = \begin{bmatrix} \frac{1}{15} & 0 \\ 0 & \frac{1}{15} \end{bmatrix}$$

$$\mathbf{X}^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}.$$

Given data: $\sigma^2 = -3$ dB $\implies 10\log\sigma^2 = -3 \implies \sigma^2 = \frac{1}{2}$ and $\sigma_h^2 = 0$ dB $\implies 10\log\sigma^2 = 0 \implies \sigma^2 = 1$.
The LMMSE estimate of $\mathbf{h}$ is given by,

$$\begin{aligned}\hat{\mathbf{h}} &= \sigma_h^2(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I})^{-1}\mathbf{X}^T\mathbf{y} \\ &= \begin{bmatrix} 16/15 \\ -4/15 \end{bmatrix}.\end{aligned}$$

Ans (d)

6. The ML estimate of $\mathbf{h}$ for a multi-antenna channel estimation scenario is given by,

$$\hat{\mathbf{h}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}.$$

From the solution to above problem, we obtain

$$\hat{\mathbf{h}} = \begin{bmatrix} 8/7 \\ -2/7 \end{bmatrix}.$$

Ans (b)

7. The error covariance of the LMMSE estimate of $\mathbf{h}$ for a multi-antenna channel estimation scenario is given by,

$$\mathrm{E}\{(\hat{\mathbf{h}} - \mathbf{h})(\hat{\mathbf{h}} - \mathbf{h})^T\} = \left(\frac{\mathbf{X}^T\mathbf{X}}{\sigma^2} + \frac{\mathbf{I}}{\sigma_h^2}\right)^{-1}.$$

From the solution to problem 5, we obtain

$$\mathrm{E}\{(\hat{\mathbf{h}} - \mathbf{h})(\hat{\mathbf{h}} - \mathbf{h})^T\} = \begin{bmatrix} \frac{1}{15} & 0 \\ 0 & \frac{1}{15} \end{bmatrix}.$$

Ans (a)

8. The error covariance of the LMMSE estimate of $\mathbf{h}$ for a multi-antenna channel estimation scenario is given by,

$$\mathrm{E}\{(\hat{\mathbf{h}}-\mathbf{h})(\hat{\mathbf{h}}-\mathbf{h})^T\} = \left(\frac{\mathbf{X}^T\mathbf{X}}{\sigma^2} + \frac{\mathbf{I}}{\sigma_h^2}\right)^{-1}.$$

When $\sigma_h^2\mathbf{X}^T\mathbf{X} >> \sigma^2\mathbf{I}$, the error covariance of the LMMSE estimate of $\mathbf{h}$ reduces to,

$$\begin{aligned}\mathrm{E}\{(\hat{\mathbf{h}}-\mathbf{h})(\hat{\mathbf{h}}-\mathbf{h})^T\} &= \left(\frac{\mathbf{X}^T\mathbf{X}}{\sigma^2}\right)^{-1} \\ &= \begin{bmatrix}\frac{1}{14} & 0 \\ 0 & \frac{1}{14}\end{bmatrix}.\end{aligned}$$

Ans (a)

9. For a multi-antenna channel estimation scenario with $N = 4$ pilot vectors,

$$\mathbf{x}(1) = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{x}(2) = \begin{bmatrix}1\\-2\end{bmatrix}, \mathbf{x}(3) = \begin{bmatrix}2\\1\end{bmatrix}, \mathbf{x}(4) = \begin{bmatrix}1\\-1\end{bmatrix}.$$

The corresponding pilot matrix $\mathbf{X}$ is given by,

$$\mathbf{X} = \begin{bmatrix}\mathbf{x}(1)^T\\ \mathbf{x}(2)^T\\ \mathbf{x}(3)^T\\ \mathbf{x}(4)^T\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -2\\ 2 & 1\\ 1 & 1\end{bmatrix}.$$

The received output vectors are,

$$\mathbf{y}(1) = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \mathbf{y}(2) = \begin{bmatrix}2\\-2\\1\end{bmatrix}, \mathbf{y}(3) = \begin{bmatrix}-2\\1\\-2\end{bmatrix}, \mathbf{y}(4) = \begin{bmatrix}1\\2\\-1\end{bmatrix}.$$

Therefore, the observation matrix $\mathbf{Y}$ is,

$$\mathbf{Y} = \begin{bmatrix}\mathbf{y}(1)^T\\ \mathbf{y}(2)^T\\ \mathbf{y}(3)^T\\ \mathbf{y}(4)^T\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\ 2 & -2 & 1\\ -2 & 1 & -2\\ 1 & 2 & -1\end{bmatrix}.$$

The MIMO system model is,

$$\mathbf{Y} = \mathbf{X}\mathbf{H} + \mathbf{V}.$$

Here, size of $\mathbf{Y}$ matrix is 4 x 3 and size of $\mathbf{X}$ matrix is 4 x 2. Therefore, size of $\mathbf{H}$ matrix or size of the MIMO system is 2 x 3.
Ans (c)

10. The size of $\mathbf{H}$ matrix 2 X 3, i.e. M=2 and R=3 and the size of $\mathbf{V}$ matrix is 4 X 3. The channel coefficients are IID Gaussian with dB variance $\sigma_h^2 = 0$ dB $= 1$ and the noise samples

are also IID Gaussian with dB noise variance $\sigma^2 = -3$ dB $= \frac{1}{2}$. Therefore, the covariance matrices $\mathbf{R}_{HH}$ of the channel matrix and $\mathbf{R}_{VV}$ of the noise matrix, respectively are given by,

$$\begin{aligned}\mathbf{R}_{HH} &= \mathrm{E}\{\mathbf{H}\mathbf{H}^T\}\\ &= \mathrm{R}\sigma_h^2\mathbf{I}_{2X2}\\ &= 3\mathbf{I}_{2X2}\\ \mathbf{R}_{VV} &= \mathrm{E}\{\mathbf{V}\mathbf{V}^T\}\\ &= \mathrm{R}\sigma^2\mathbf{I}_{4X4}\\ &= \frac{3}{2}\mathbf{I}_{4X4}.\end{aligned}$$

Ans (a)