Assignment 4 (Solutions)
NPTEL MOOC (Bayesian/ MMSE Estimation for MIMO/OFDM Wireless Communications)

1. The system model can be written as,

$$\mathbf{y} = h\mathbf{x} + \mathbf{v}$$

The MSE of the MMSE estimate $\hat{h}$ of the above mentioned system model is given by,

$$\begin{aligned}
\mathrm{E}\{|\hat{h} - h|^2\} &= r_{hh} - \mathbf{r}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{r}_{yh} \\
&= \sigma_h^2 - \sigma_h^2\mathbf{x}^H(\sigma_h^2\mathbf{x}\mathbf{x}^H + \sigma^2\mathbf{I})^{-1}\mathbf{x}\sigma_h^2 \\
&= \sigma_h^2 - \frac{\sigma_h^4||\mathbf{x}||^2}{\sigma_h^2||\mathbf{x}||^2 + \sigma^2} \\
&= \frac{1}{\frac{1}{\sigma^2/||\mathbf{x}||^2} + \frac{1}{\sigma_h^2}}. \qquad (1)
\end{aligned}$$

Given data: $\mu_h = 1 + j,\ \sigma_h^2 = 1/2,\ N = 4,\ \sigma^2 = 3\ dB \implies 10\log\sigma^2 = 3 \implies \sigma^2 \approx 2$

$$\mathbf{x} = \begin{bmatrix} 2+j \\ -1-j \\ 1-2j \\ -1+j \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2+2j \\ -j \\ -2+j \\ 1-j \end{bmatrix},$$

$$\begin{aligned}
||\mathbf{x}||^2 &= |2+j|^2 + |-1-j|^2 + |1-2j|^2 + |-1+j|^2 \\
&= 4+1+1+1+1+4+1+1 \\
&= 14.
\end{aligned}$$

Substituting all the values in equation (1), we get

$$\begin{aligned}
\mathrm{E}\{|\hat{h} - h|^2\} &= \frac{1}{\frac{1}{2/14} + \frac{1}{1/2}} \\
&= \frac{1}{9}.
\end{aligned}$$

Ans (a)

2. Refer to the notes of week 3 for this question.
MMSE estimate of the complex fading coefficient h is given by,

$$\hat{h} = \hat{h}_R + j\hat{h}_I.$$

From the solution of problem 1, the MSEs of the real, imaginary parts of $\hat{h}$ can be obtained as,

$$\begin{aligned}
&\text{MSE of the real part of } \hat{h} = \text{MSE of the imaginary part of } \hat{h} \\
&= \mathrm{E}\{|\hat{h}_R - h_R|^2\} = \mathrm{E}\{|\hat{h}_I - h_I|^2\} \\
&= \frac{1}{2}\mathrm{E}\{|\hat{h} - h|^2\} = \frac{1}{2}\left(\frac{1}{\frac{1}{\sigma^2/||\mathbf{x}||^2} + \frac{1}{\sigma_h^2}}\right) \\
&= \frac{1}{18}.
\end{aligned}$$

Ans(c)

3. Let h_R denotes the real part of the true parameter h and $\hat{h}_R$ be the real part of the estimate $\hat{h}$. Further, $\hat{h}_R - h_R$ gives the estimation error in the real part of the estimate. Also, from the solutions to problem 1 and 2 we can say, $h_R \sim \mathcal{N}\big(\hat{h}_R, \frac{1}{18}\big)$. Therefore, $h_R - \hat{h}_R$ is distributed as a zero-mean Gaussian with variance 1/18.
Hence, $\hat{h}_R - h_R \sim \mathcal{N}\big(0, \frac{1}{18}\big)$.
Further, $\frac{\hat{h}_R - h_R}{\sqrt{\frac{1}{18}}}$ is a zero-mean unit-variance Gaussian RV. Probability that the real part of the MMSE estimate $\hat{h}$ lies within a radius 1/2 of the unknown parameter h can be calculated as follows,

$$\begin{aligned}
&\Pr\left(|\hat{h}_R - h_R| \leq \frac{1}{2}\right) = \Pr\left(\frac{|\hat{h}_R - h_R|}{\sqrt{\frac{1}{18}}} \leq \frac{\frac{1}{2}}{\sqrt{\frac{1}{18}}}\right) = 1 - \Pr\left(\frac{|\hat{h}_R - h_R|}{\sqrt{\frac{1}{18}}} \geq \frac{\frac{1}{2}}{\sqrt{\frac{1}{18}}}\right) \\
&= 1 - \left\{\Pr\left(\frac{\hat{h}_R - h_R}{\sqrt{\frac{1}{18}}} \geq \frac{\frac{1}{2}}{\sqrt{\frac{1}{18}}}\right) + \Pr\left(\frac{\hat{h}_R - h_R}{\sqrt{\frac{1}{18}}} \leq -\frac{\frac{1}{2}}{\sqrt{\frac{1}{18}}}\right)\right\} \\
&= 1 - 2\Pr\left(\frac{\hat{h}_R - h_R}{\sqrt{\frac{1}{18}}} \geq \frac{\frac{1}{2}}{\sqrt{\frac{1}{18}}}\right) = 1 - 2Q\left(\frac{\frac{1}{2}}{\sqrt{\frac{1}{18}}}\right) = 1 - 2Q\left(\sqrt{\frac{9}{2}}\right)
\end{aligned}$$

Further, since the errors in the real and imaginary parts are independent as they are Gaussian, the probability that both the real and imaginary parts of the MMSE estimate $\hat{h}$ lie within a radius of 1/2 from the real and imaginary parts of the unknown parameter h respectively is $\left(1 - 2Q\left(\sqrt{\frac{9}{2}}\right)\right)^2$.

Ans (b)

4. To estimate the unknown parameter h, we have each observation as

$$y(k) = h + v(k), \text{for } 1 \leq k \leq N,$$

where $v(k) \sim \mathcal{N}(0,\, \sigma_k^2)$, $h \sim \mathcal{N}(\mu_h,\, \sigma_h^2)$. By stacking N such observations, we obtain observation vector as

$$\mathbf{y} = \mathbf{1}h + \mathbf{v},$$

where mean of the noise vector is $\mathrm{E}\{\mathbf{v}\} = \mathbf{0}$ and its covariance matrix is denoted by $\mathbf{C}_v = \mathrm{E}\{\mathbf{v}\mathbf{v}^T\}$. So, the mean of the observation vector is denoted by $\mu_y = \mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{\mathbf{1}h + \mathbf{v}\} = \mathbf{1}\mu_h$ and the observation covariance matrix can be calculated as

$$\begin{aligned}
\mathbf{R}_{yy} &= \mathrm{E}\{(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^T\} \\
&= \mathrm{E}\{(\mathbf{1}(h - \mu_h) + \mathbf{v})(\mathbf{1}(h - \mu_h) + \mathbf{v})^T\} \\
&= \mathrm{E}\{(h - \mu_h)^2\}\mathbf{1}\mathbf{1}^T + \mathrm{E}\{\mathbf{v}\mathbf{v}^T\} + \mathbf{1}\mathrm{E}\{(h - \mu_h)\mathbf{v}^T\} + \mathrm{E}\{\mathbf{v}(h - \mu_h)\}\mathbf{1}^T \\
&= \sigma_h^2\mathbf{1}\mathbf{1}^T + \mathbf{C}_v.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{R}_{hy} &= \mathrm{E}\{(h - \mu_h)(\mathbf{y} - \mu_y)^T\} \\
&= \mathrm{E}\{(h - \mu_h)(\mathbf{1}(h - \mu_h) + \mathbf{v})^T\} \\
&= \mathrm{E}\{(h - \mu_h)^2\}\mathbf{1}^T + \mathrm{E}\{(h - \mu_h)\mathbf{v})^T\} \\
&= \sigma_h^2\mathbf{1}^T.
\end{aligned}$$

The MMSE estimate of the unknown parameter h is given by,

$$\hat{h} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}(\mathbf{y} - \mu_y) + \mu_h.$$

Substituting the values of the covariance matrices in the above expression, we obtain

$$\hat{h} = \sigma_h^2\mathbf{1}^T(\sigma_h^2\mathbf{1}\mathbf{1}^T + \mathbf{C}_v)^{-1}(\mathbf{y} - \mathbf{1}\mu_h) + \mu_h. \tag{2}$$

Simplifying the above expression using Woodbury matrix identity, we get

$$\begin{aligned}
\sigma_h^2\mathbf{1}^T(\sigma_h^2\mathbf{1}\mathbf{1}^T+\mathbf{C}_v)^{-1} &= \sigma_h^2\mathbf{1}^T\Big(\mathbf{C}_v^{-1}-\mathbf{C}_v^{-1}\mathbf{1}\Big(\frac{1}{\sigma_h^2}+\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\Big)^{-1}\mathbf{1}^T\mathbf{C}_v^{-1}\Big)\\
&= \sigma_h^2\mathbf{1}^T\mathbf{C}_v^{-1}-\sigma_h^2\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\Big(\frac{1}{\sigma_h^2}+\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\Big)^{-1}\mathbf{1}^T\mathbf{C}_v^{-1}\\
&= \sigma_h^2\mathbf{1}^T\mathbf{C}_v^{-1}-\sigma_h^2\Big(\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}+\frac{1}{\sigma_h^2}-\frac{1}{\sigma_h^2}\Big)\Big(\frac{1}{\sigma_h^2}+\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\Big)^{-1}\mathbf{1}^T\mathbf{C}_v^{-1}\\
&= \Big(\frac{1}{\sigma_h^2}+\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\Big)^{-1}\mathbf{1}^T\mathbf{C}_v^{-1}.
\end{aligned}$$

After substituting the above expression in equation (2), we obtain

$$\hat{h}=\Big(\frac{1}{\sigma_h^2}+\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\Big)^{-1}\mathbf{1}^T\mathbf{C}_v^{-1}(\mathbf{y}-\mathbf{1}\mu_h)+\mu_h. \tag{3}$$

Simplifying $\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}$, we get

$$\mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}=\begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sigma_1^2} & 0 & \dots & 0\\ 0 & \frac{1}{\sigma_2^2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \frac{1}{\sigma_N^2}\end{bmatrix}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}=\sum_{k=1}^{N}\frac{1}{\sigma_k^2}.$$

Similarly,

$$\mathbf{1}^T\mathbf{C}_v^{-1}(\mathbf{y}-\mathbf{1}\mu_h)=\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\,\big(y(k)-\mu_h\big).$$

Now, equation (3) can be written as,

$$\hat{h}=\mu_h+\frac{\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\,\big(y(k)-\mu_h\big)}{\frac{1}{\sigma_h^2}+\sum_{k=1}^{N}\frac{1}{\sigma_k^2}}=\frac{\sum_{k=1}^{N}\frac{y(k)}{\sigma_k^2}+\frac{\mu_h}{\sigma_h^2}}{\sum_{k=1}^{N}\frac{1}{\sigma_k^2}+\frac{1}{\sigma_h^2}}.$$

Ans (d)

5. The system model can be written as,

$$\mathbf{y}=\mathbf{1}h+\mathbf{v}.$$

The MSE of the MMSE estimate $\hat{h}$ of the unknown Gaussian parameter h is given by,

$$\mathrm{E}\{(\hat{h}-h)^2\}=r_{hh}-\mathbf{r}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{r}_{yh} \tag{4}$$

From the solution of problem 4, $\mathbf{R}_{yy}, \mathbf{r}_{hy}$ can be written as

$$\mathbf{R}_{yy} = \sigma_h^2 \mathbf{1}\mathbf{1}^T + \mathbf{C}_v,$$
$$\mathbf{r}_{hy} = \sigma_h^2 \mathbf{1}^T.$$

Substituting the values, equation (4) can be written as

$$\begin{aligned}
\mathrm{E}\{(\hat{h}-h)^2\} &= \sigma_h^2 - \sigma_h^2 \mathbf{1}^T(\sigma_h^2 \mathbf{1}\mathbf{1}^T + \mathbf{C}_v)^{-1}\mathbf{1}\sigma_h^2 \\
&= \sigma_h^2 - \left(\frac{1}{\sigma_h^2} + \mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\right)^{-1} \mathbf{1}^T\mathbf{C}_v^{-1}\mathbf{1}\sigma_h^2 \\
&= \sigma_h^2 - \frac{\sum_{k=1}^N \frac{\sigma_h^2}{\sigma_k^2}}{\frac{1}{\sigma_h^2} + \sum_{k=1}^N \frac{1}{\sigma_k^2}} \\
&= \frac{1}{\frac{1}{\sigma_h^2} + \sum_{k=1}^N \frac{1}{\sigma_k^2}} \\
&= \left(\frac{1}{\sigma_h^2} + \sum_{k=1}^N \frac{1}{\sigma_k^2}\right)^{-1}.
\end{aligned}$$

Ans (a)

6. The LMMSE estimate is identical to the MMSE estimate for a Gaussian parameter.
   Ans (c)

7. As derived in the lectures notes of week 4, the LMMSE estimate $\hat{h}$ of the unknown parameter h, which is not necessarily Gaussian is given by

$$\hat{h} = \mathbf{r}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{y}.$$

   Ans (d)

8. In this scenario, we have $N = 4$ pilot vectors, each corresponding to M transmit antennas. The length of each pilot vector is 2. Hence, $M = 2$.
   Ans (b)

9. For a multi-antenna channel estimation scenario with $N = 4$ pilot vectors

$$\mathbf{x}(1) = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{x}(2) = \begin{bmatrix}1\\-2\end{bmatrix}, \mathbf{x}(3) = \begin{bmatrix}2\\1\end{bmatrix}, \mathbf{x}(4) = \begin{bmatrix}1\\-1\end{bmatrix}.$$

The corresponding pilot matrix $\mathbf{X}$ is given by,

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}(1)^T \\ \mathbf{x}(2)^T \\ \mathbf{x}(3)^T \\ \mathbf{x}(4)^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Ans (d)

10. From the solution of problem 9, the pilot matrix $\mathbf{X}$ can be written as,

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}.$$

Columns $\mathbf{c}_1$ and $\mathbf{c}_2$ satisfy the orthogonality property, i.e. $\mathbf{c}_1^T \mathbf{c}_2 = 0$. Hence, the pilot matrix $\mathbf{X}$ has orthogonal columns.
Ans (c)