Assignment-3 Solutions

1. Consider MMSE estimation for the wireless sensor network (WSN) scenario as described in lectures with each observation $y(k) = h + v(k)$, for $1 \leq k \leq N$, i.e. number of observations N and IID Gaussian noise samples of variance σ^2. The expression for the MSE (Mean Squared Error) of the ML estimate of the unknown Gaussian parameter h is $\frac{\sigma^2}{N}$
Ans-(a)

2. Minimum Mean Squared Error(MMSE) in estimating $\hat{h}$ is defined as,

$$\mathrm{E}\{(\hat{h} - h)^2\} = r_{hh} - \mathbf{r}_{hy} R_{yy}^{-1} \mathbf{r}_{yh} \tag{1}$$

Considering wireless sensor network,

$$\mathbf{y} = \mathbf{1}h + \mathbf{v} \tag{2}$$

with $h \sim N(\mu_h, \sigma_h^2)$

$$\begin{aligned}
r_{hh} &= \mathrm{E}\{(\hat{h} - h)^2\} = \sigma_h^2 \\
\mathbf{r}_{hy} &= \mathrm{E}\{(h - \mu_h)(\mathbf{y} - \mu_y)^T = \sigma_h^2 \mathbf{1}^T = \mathbf{r}_{yh}^T \\
\mathbf{R}_{yy} &= \mathrm{E}\{(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^T = \sigma_h^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I} \\
\mathrm{E}\{(\hat{h} - h)^2\} &= \sigma_h^2 - \sigma_h^2 \mathbf{1}^T (\sigma_h^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})^{-1} \mathbf{1} \sigma_h^2 \\
&= \sigma_h^2 - \frac{\sigma_h^4 N}{(\sigma_h^2 N + \sigma^2)} \\
&= \frac{\sigma_h^2 \sigma^2}{(\sigma_h^2 N + \sigma^2)} \\
\mathrm{E}\{(\hat{h} - h)^2\} &= \frac{1}{\frac{1}{\frac{\sigma^2}{N}} + \frac{1}{\sigma_h^2}}
\end{aligned}$$

Ans-(b)

3. Given, $N = 5$, $\mu_h = 1/2$, $\sigma_h^2 = 1/4$, $\sigma^2 = -3dB = 1/2$
MSE for ML estimate=

$$\begin{aligned}
&= \frac{\sigma^2}{N} \\
&= \frac{1/2}{5} \\
&= \frac{1}{10}
\end{aligned}$$

Ans-(b)

4. Given, $\sigma^2 = -3dB = 1/2$, $\sigma_h^2 = 1/4$

$$\begin{aligned}
P\left(\left|\hat{h} - h\right| < \frac{\sigma}{4}\right) &> 0.999 \\
P\left(\left|\hat{h} - h\right| \geq \frac{\sigma}{4}\right) &\leq 10^{-3}
\end{aligned}$$

let $w = \left|\hat{h} - h\right|$

$$
\begin{aligned}
w = (\hat{h} - h) &\sim N\left(0, \frac{1}{\frac{1}{\frac{\sigma^2}{N}} + \frac{1}{\sigma_h^2}}\right) \\
\tilde{\sigma}^2 &= \frac{1}{\frac{1}{\frac{\sigma^2}{N}} + \frac{1}{\sigma_h^2}} (let) \\
P\left(|w| \geq \frac{\sigma}{4}\right) &\leq 10^{-3} \\
2P\left(w \geq \frac{\sigma}{4}\right) &\leq 10^{-3} \\
P\left(w \geq \frac{\sigma}{4}\right) &\leq 5 \times 10^{-4} \\
P\left(\frac{w}{\tilde{\sigma}} \geq \frac{\sigma/4}{\tilde{\sigma}}\right) &\leq 5 \times 10^{-4} \\
Q\left(\frac{\sigma/4}{\tilde{\sigma}}\right) &\leq 5 \times 10^{-4} \\
\left(\frac{\sigma/4}{\tilde{\sigma}}\right) &\geq Q^{-1}\left(5 \times 10^{-4}\right) \\
\frac{\sigma}{\frac{1}{\sqrt{\frac{1}{\frac{\sigma^2}{N}} + \frac{1}{\sigma_h^2}}}} &\geq 4Q^{-1}\left(5 \times 10^{-4}\right) \\
\sigma\sqrt{\left(\frac{N}{\sigma^2} + \frac{1}{\sigma_h^2}\right)} &\geq 4Q^{-1}\left(5 \times 10^{-4}\right) \\
\sqrt{\left(N + \frac{\sigma^2}{\sigma_h^2}\right)} &\geq 4Q^{-1}\left(5 \times 10^{-4}\right) \\
N + \frac{\sigma^2}{\sigma_h^2} &\geq \left(4Q^{-1}\left(5 \times 10^{-4}\right)\right)^2 \\
N + \frac{1/2}{1/4} &\geq \left(4Q^{-1}\left(5 \times 10^{-4}\right)\right)^2 \\
N &\geq \left(4Q^{-1}\left(5 \times 10^{-4}\right)\right)^2 - 2 \\
N &\geq (4 \times 3.2905)^2 - 2 \\
N &\geq 171.2411 N_{min} = 172
\end{aligned}
$$

Ans-(a)

5. For fading channel estimation problem, where the output symbol $y(k)$ is $y(k) = hx(k) + v(k)$, with h, $x(k)$, $v(k)$ denoting the real channel coefficient, pilot symbol and noise sample respectively, the expression for the MSE (Mean Squared Error) of the ML estimate $\hat{h}$ of the unknown parameter h is, $\frac{\sigma^2}{\|\mathbf{x}\|^2}$
Ans-(d)

6. For fading channel estimation problem, MMSE is,

$$
\begin{aligned}
\mathrm{E}\{(\hat{h} - h)^2\} &= r_{hh} - \mathbf{r}_{hy} R_{yy}^{-1} \mathbf{r}_{yh} \\
r_{hh} &= \mathrm{E}\{(\hat{h} - h)^2\} = \sigma_h^2 \\
\mathbf{r}_{hy} &= \mathrm{E}\{(h - \mu_h)(\mathbf{y} - \mu_y)^T = \sigma_h^2 \mathbf{x}^T = \mathbf{r}_{yh}^T \\
\mathbf{R}_{yy} &= \mathrm{E}\{(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^T = \sigma_h^2 \mathbf{1}\mathbf{x}^T + \sigma^2 \mathbf{I} \\
\mathrm{E}\{(\hat{h} - h)^2\} &= \sigma_h^2 - \sigma_h^2 \mathbf{x}^T (\sigma_h^2 \mathbf{x}\mathbf{x}^T + \sigma^2 \mathbf{I})^{-1} \mathbf{x} \sigma_h^2
\end{aligned}
$$

we know that,

$$
\sigma_h^2 \mathbf{x}^T (\sigma_h^2 \mathbf{x}\mathbf{x}^T + \sigma^2 \mathbf{I})^{-1} = (\sigma_h^2 \mathbf{x}^T \mathbf{x} + \sigma^2)^{-1} \sigma_h^2 \mathbf{x}^T
$$

substituting this in above expression gives,

$$\begin{aligned}
\mathrm{E}\{(\hat{h}-h)^2\} &= \sigma_h^2 - (\sigma_h^2\mathbf{x}^T\mathbf{x}+\sigma^2)^{-1}\sigma_h^2\mathbf{x}^T\mathbf{x}\sigma_h^2 \\
&= \sigma_h^2 - \frac{\sigma_h^2\mathbf{x}^T\mathbf{x}\sigma_h^2}{(\sigma_h^2\mathbf{x}^T\mathbf{x}+\sigma^2)} \\
&= \frac{\sigma^2\sigma_h^2}{(\sigma_h^2\mathbf{x}^T\mathbf{x}+\sigma^2)} \\
&= \frac{1}{\frac{1}{\frac{\sigma^2}{\mathbf{x}^T\mathbf{x}}}+\frac{1}{\sigma_h^2}} \\
&= \frac{1}{\frac{1}{\frac{\sigma^2}{\|\mathbf{x}\|}}+\frac{1}{\sigma_h^2}}
\end{aligned}$$

Ans-(b)

7. We know that,

$$\text{MMSE} = \frac{1}{\frac{1}{\text{MSE of ML}}+\frac{1}{\text{Prior Var}}}$$

As the number of samples N becomes very large, the ML estimate becomes more accurate, and hence, MSE of ML<<<Prior Variance. which imply

$\frac{1}{\text{MSE of ML}} >>> \frac{1}{\text{Prior Var}}$

hence, MMSE becomes,

$$\text{MMSE = MSE of ML = } \frac{\sigma^2}{\|\mathbf{x}\|^2}$$

Ans-(b)

8. Given $\mathbf{x} = \begin{bmatrix} 1/2 \\ 2 \\ 1 \\ 3/2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \end{bmatrix}$, $\sigma^2 = 3dB = 2$, $\sigma_h^2 = 1/2$, $\mu_h = 1$

$$\begin{aligned}
\text{MSE for ML estimate} &= \frac{\sigma^2}{\|\mathbf{x}\|} \\
&= \frac{2}{(1/2)^2+(2)^2+(1)^2+(3/2)^2} \\
&= \frac{4}{15}
\end{aligned}$$

$$\begin{aligned}
\text{MMSE} &= \frac{1}{\frac{1}{\frac{\sigma^2}{\|\mathbf{x}\|^2}}+\frac{1}{\sigma_h^2}} \\
&= \frac{1}{\frac{1}{\frac{2}{15/2}}+\frac{1}{1/2}} \\
&= \frac{1}{\frac{15}{4}+\frac{2}{1}} \\
&= \frac{4}{23}
\end{aligned}$$

Ans-(a)

9. True parameter h to be within $\sigma/8$ of the estimate $\hat{h}$ with probability greater than 0.9999,

$$P\left(\left|\hat{h}-h\right| < \frac{\sigma}{8}\right) > 0.9999$$
$$P\left(\left|\hat{h}-h\right| \geq \frac{\sigma}{8}\right) \leq 10^{-4}$$

let $w = \left|\hat{h}-h\right|$ also, we have $|x(k)|^2 = 4$ for all k.

$$w = (h-\hat{h}) \sim N\left(0, \frac{1}{\frac{1}{\frac{\sigma^2}{\|\mathbf{x}\|^2}} + \frac{1}{\sigma_h^2}}\right)$$
$$\tilde{\sigma}^2 = \frac{1}{\frac{1}{\frac{\sigma^2}{\|\mathbf{x}\|^2}} + \frac{1}{\sigma_h^2}} (let)$$
$$P\left(|w| \geq \frac{\sigma}{8}\right) \leq 10^{-4}$$
$$2P\left(w \geq \frac{\sigma}{8}\right) \leq 10^{-4}$$
$$P\left(w \geq \frac{\sigma}{8}\right) \leq 5\times 10^{-5}$$
$$P\left(\frac{w}{\tilde{\sigma}} \geq \frac{\sigma/8}{\tilde{\sigma}}\right) \leq 5\times 10^{-5}$$
$$Q\left(\frac{\sigma/8}{\tilde{\sigma}}\right) \leq 5\times 10^{-5}$$
$$\left(\frac{\sigma/8}{\tilde{\sigma}}\right) \geq Q^{-1}\left(5\times 10^{-5}\right)$$
$$\frac{\sigma}{\frac{1}{\sqrt{\frac{1}{\frac{\sigma^2}{\|\mathbf{x}\|^2}} + \frac{1}{\sigma_h^2}}}} \geq 8Q^{-1}\left(5\times 10^{-5}\right)$$
$$\sigma\sqrt{\left(\frac{\|\mathbf{x}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}\right)} \geq 8Q^{-1}\left(5\times 10^{-5}\right)$$
$$\sqrt{\left(4N + \frac{\sigma^2}{\sigma_h^2}\right)} \geq 8Q^{-1}\left(5\times 10^{-5}\right)$$
$$4N + \frac{\sigma^2}{\sigma_h^2} \geq \left(8Q^{-1}\left(5\times 10^{-5}\right)\right)^2$$
$$4N \geq \left(8Q^{-1}\left(5\times 10^{-5}\right)\right)^2 - \frac{\sigma^2}{\sigma_h^2}$$
$$N \geq \frac{1}{4}\left(\left(8Q^{-1}\left(5\times 10^{-5}\right)\right)^2 - \frac{\sigma^2}{\sigma_h^2}\right)$$
$$N_{min} = \frac{1}{4}\left(\left(8Q^{-1}\left(5\times 10^{-5}\right)\right)^2 - \frac{\sigma^2}{\sigma_h^2}\right)$$

Ans-(d)

10. Given $\mathbf{x} = \begin{bmatrix} 2+j \\ -1-j \\ 1-2j \\ -1+j \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2+2j \\ -j \\ -2+j \\ 1-j \end{bmatrix}$, $\sigma^2 = 3dB = 2$, $\sigma_h^2 = 1/2$, $\mu_h = 1+j$

$$\|\mathbf{x}\|^2 = |2+j|^2 + |-1-j|^2 + |1-2j|^2 + |-1+j|^2$$
$$= (5) + (2) + (5) + (2) = 14$$

$$\text{MMSE Esimate} = \frac{\frac{\mathbf{x}^T\mathbf{y}}{\sigma^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\frac{\sigma^2}{\|\mathbf{x}\|^2}} + \frac{1}{\sigma_h^2}}$$
$$= \frac{\frac{1}{2} + \frac{1+j}{1/2}}{\frac{1}{\frac{2}{14}} + \frac{1}{1/2}}$$
$$= \frac{\frac{5}{2} + 2j}{\frac{7}{1} + \frac{2}{1}}$$
$$= \frac{5}{18} + \frac{2}{9}j$$

Ans-(c)