Assignment 2 (Solutions)
NPTEL MOOC (Bayesian/ MMSE Estimation for MIMO/OFDM Wireless Communications)

1. To estimate the unknown parameter h, we have each observation as

$$y(k) = h + v(k), \text{for } 1 \leq k \leq N,$$

where $v(k) \sim \mathcal{N}(0,\ \sigma^2)$, h is a Gaussian parameter with mean as $\mathrm{E}\{h\} = \mu_h$ and variance as $\mathrm{E}\{(h-\mu_h)^2\} = \sigma_h^2$. By stacking N such observations, we obtain observation vector as

$$\mathbf{y} = \mathbf{1}h + \mathbf{v},$$

where mean of the noise vector is $\mathrm{E}\{\mathbf{v}\} = \mathbf{0}$ and its covariance matrix is denoted by $\mathrm{E}\{\mathbf{v}\mathbf{v}^T\} = \sigma^2\mathbf{I}$. So, the mean of the observation vector is denoted by $\mu_y = \mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{\mathbf{1}h + \mathbf{v}\} = \mathbf{1}\mu_h$ and the observation covariance matrix can be calculated as

$$\begin{aligned}\mathbf{R}_{yy} &= \mathrm{E}\{(\mathbf{y}-\mu_y)(\mathbf{y}-\mu_y)^T\} \\ &= \mathrm{E}\{(\mathbf{1}(h-\mu_h)+\mathbf{v})(\mathbf{1}(h-\mu_h)+\mathbf{v})^T\} \\ &= \mathrm{E}\{(h-\mu_h)^2\}\mathbf{1}\mathbf{1}^T + \mathrm{E}\{\mathbf{v}\mathbf{v}^T\} + \mathbf{1}\mathrm{E}\{(h-\mu_h)\mathbf{v}^T\} + \mathrm{E}\{\mathbf{v}(h-\mu_h)\}\mathbf{1}^T \\ &= \sigma_h^2\mathbf{1}\mathbf{1}^T + \sigma^2\mathbf{I}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{R}_{hy} &= \mathrm{E}\{(h-\mu_h)(\mathbf{y}-\mu_y)^T\} \\ &= \mathrm{E}\{(h-\mu_h)(\mathbf{1}(h-\mu_h)+\mathbf{v})^T\} \\ &= \mathrm{E}\{(h-\mu_h)^2\}\mathbf{1}^T + \mathrm{E}\{(h-\mu_h)\mathbf{v})^T\} \\ &= \sigma_h^2\mathbf{1}^T.\end{aligned}$$

The MMSE estimate of the unknown parameter h is given by

$$\hat{h} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}(\mathbf{y}-\mu_y) + \mu_h.$$

Substituting the values of the covariance matrices in the above expression, we obtain

$$\hat{h} = \sigma_h^2 \mathbf{1}^T(\sigma_h^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})^{-1}(\mathbf{y} - \mathbf{1}\mu_h) + \mu_h. \tag{1}$$

Simplifying the above expression, we get

$$\begin{aligned}
\sigma_h^2 \mathbf{1}^T(\sigma_h^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})^{-1} &= (\sigma_h^2 \mathbf{1}^T\mathbf{1} + \sigma^2)^{-1}\sigma_h^2 \mathbf{1}^T \\
&= (\sigma_h^2 N + \sigma^2)^{-1}\sigma_h^2 \mathbf{1}^T \\
&= \frac{\sigma_h^2 \mathbf{1}^T}{\sigma_h^2 N + \sigma^2}.
\end{aligned}$$

After substitution in equation (1), we obtain

$$\begin{aligned}
\hat{h} &= \frac{\sigma_h^2 \mathbf{1}^T}{(\sigma_h^2 N + \sigma^2)}(\mathbf{y} - \mathbf{1}\mu_h) + \mu_h \\
&= \frac{\sigma_h^2 \mathbf{1}^T \mathbf{y}}{\sigma_h^2 N + \sigma^2} - \frac{\sigma_h^2 N \mu_h}{\sigma_h^2 N + \sigma^2} + \mu_h \\
&= \frac{\sigma_h^2 \mathbf{1}^T \mathbf{y} + \sigma^2 \mu_h}{\sigma_h^2 N + \sigma^2} \\
&= \frac{\frac{\mathbf{1}^T \mathbf{y}}{\sigma^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{N}{\sigma^2} + \frac{1}{\sigma_h^2}} \\
&= \frac{\frac{(\mathbf{1}^T \mathbf{y})/N}{\sigma^2/N} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/N} + \frac{1}{\sigma_h^2}}.
\end{aligned}$$

Ans (b)

2. As evaluated in Q1, the MMSE estimate can be written as

$$\begin{aligned}
\hat{h} &= \frac{\frac{(\mathbf{1}^T \mathbf{y})/N}{\sigma^2/N} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/N} + \frac{1}{\sigma_h^2}} \\
&= \frac{\frac{\text{MLE}}{\text{Var of MLE}} + \frac{\text{Prior Mean}}{\text{Prior Var}}}{\frac{1}{\text{Var of MLE}} + \frac{1}{\text{Prior Var}}}.
\end{aligned}$$

As $N \to \infty$, Var of MLE $\to 0$ or $\sigma^2/N << \sigma_h^2$,

$$\hat{h} \to \frac{\frac{\text{MLE}}{\text{var of MLE}}}{\frac{1}{\text{Var of MLE}}} = \text{ML Estimate.}$$

Ans (c)

3. Maximum likelihood estimate of the unknown parameter h for the model $\mathbf{y} = \mathbf{1}h + \mathbf{v}$ is given as,

$$\hat{h} = \frac{\mathbf{1}^T\mathbf{y}}{N} = \frac{\sum_{n=1}^{5} y(n)}{N}.$$

Given data: $y(1) = 1,\ y(2) = 1,\ y(3) = 2,\ y(4) = 3/2,\ y(5) = 5/2$ and $N = 5$. Substituting these values in the above expression, we obtain MLE as

$$\hat{h} = \frac{1+1+2+3/2+5/2}{5} = 8/5 = 1.6.$$

Ans (a)

4. Given data: $y(1) = 1,\ y(2) = 1,\ y(3) = 2,\ y(4) = 3/2,\ y(5) = 5/2,\ N = 5,\ \mu_h = 1/2,\ \sigma_h^2 = 1/4,\ \sigma^2 = -3\ dB \implies 10\log\sigma^2 = -3 \implies \sigma^2 = 1/2$. The MMSE estimate of the unknown parameter h is given as

$$\hat{h} = \frac{\frac{(\mathbf{1}^T\mathbf{y})/N}{\sigma^2/N} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/N} + \frac{1}{\sigma_h^2}}.$$

Substituting the values in the above expression we obtain

$$\hat{h} = \frac{\frac{8/5}{1/10} + \frac{1/2}{1/4}}{\frac{1}{1/10} + \frac{1}{1/4}} = \frac{18}{14} = \frac{9}{7}.$$

Ans (d)

5. Please refer to the notes of week 1 for this question.
Each observation is given by

$$y(k) = h + v(k), \text{for } 0 \le k \le 5.$$

Stacking all such N observations, we obtain the observation vector as

$$\mathbf{y} = \mathbf{1}h + \mathbf{v}.$$

Since, the noise samples are IID Gaussian and the unknown parameter h is also Gaussian, which means $\mathbf{y}$ is also Gaussian. The mean of the observation vector is $\mu_y = \mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{h\mathbf{x}+\mathbf{v}\} = \mathbf{x}\mu_h$ and the covariance matrix is given by $\mathbf{R}_{yy} = \mathrm{E}\{(\mathbf{y}-\mu_y)(\mathbf{y}-\mu_y)^T\} = \sigma_h^2\mathbf{x}\mathbf{x}^T + \sigma^2\mathbf{I}$. The posterior probability density function of the unknown parameter h can be written as

$$f_{H|Y}(h|\mathbf{y}) = \frac{f_{H,Y}(h,\mathbf{y})}{f_Y(\mathbf{y})},$$

where $f_{H,Y}(h,\mathbf{y})$ denotes the joint Gaussian distribution of h, $\mathbf{y}$ and $f_Y(\mathbf{y})$ denotes the marginal pdf of $\mathbf{y}$. So, we can say that the posterior pdf of the unknown parameter h given by $f_{H|Y}(h|\mathbf{y})$ is Gaussian.
Ans (a)

6. Given observation

$$y(k) = hx(k) + v(k),\ 0 \leq k \leq N.$$

After stacking all such N observations, received vector $\mathbf{y}$ can be written as

$$\mathbf{y} = h\mathbf{x} + \mathbf{v},$$

where $\mathrm{E}\{\mathbf{v}\} = \mathbf{0}$ and $\mathrm{E}\{\mathbf{v}\mathbf{v}^T\} = \sigma^2\mathbf{I}$. Similarly, $\mu_y = \mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{h\mathbf{x} + \mathbf{v}\} = \mathbf{x}\mu_h$ The probability density function of vector $\mathbf{y}$ can be written as

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^N\sigma^{2N}}}\exp^{\frac{-||\mathbf{y}-\mathbf{x}h||^2}{2\sigma^2}}.$$

To calculate the maximum likelihood estimator, we have to maximize the likelihood function i.e. we have to minimize the exponential term

$$\begin{aligned}||\mathbf{y} - \mathbf{x}h||^2 &= (\mathbf{y} - \mathbf{x}h)^T(\mathbf{y} - \mathbf{x}h)\\ &= \mathbf{y}^T\mathbf{y} - 2h\mathbf{x}^T\mathbf{y} + h^2\mathbf{x}^T\mathbf{x}.\end{aligned}$$

Differentiating above equation wrt h, we get ML estimate as

$$\begin{aligned}-2\mathbf{x}^T\mathbf{y} + 2h\mathbf{x}^T\mathbf{x} = 0\\ \hat{h} = \frac{\mathbf{x}^T\mathbf{y}}{||\mathbf{x}||^2}.\end{aligned}$$

Ans (c)

7. The received vector $\mathbf{y}$ can be written as

$$\mathbf{y} = h\mathbf{x} + \mathbf{v},$$

where $\mathrm{E}\{\mathbf{v}\} = \mathbf{0}$ and $\mathrm{E}\{\mathbf{v}\mathbf{v}^T\} = \sigma^2\mathbf{I}$. Similarly, $\mu_y = \mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{h\mathbf{x} + \mathbf{v}\} = \mathbf{x}\mu_h$. The covariance matrix $\mathbf{R}_{yy}$ of the output vector $\mathbf{y}$ can be calculated as

$$\begin{aligned}\mathbf{R}_{yy} &= \mathrm{E}\{(\mathbf{y}-\mu_y)(\mathbf{y}-\mu_y)^T\}\\ &= \mathrm{E}\{(\mathbf{x}(h-\mu_h)+\mathbf{v})(\mathbf{x}(h-\mu_h)+\mathbf{v})^T\}\\ &= \mathrm{E}\{(h-\mu_h)^2\}\mathbf{x}\mathbf{x}^T + \mathrm{E}\{\mathbf{v}\mathbf{v}^T\} + \mathbf{x}\mathrm{E}\{(h-\mu_h)\mathbf{v}^T\} + \mathrm{E}\{\mathbf{v}(h-\mu_h)\}\mathbf{x}^T\\ &= \sigma_h^2\mathbf{x}\mathbf{x}^T + \sigma^2\mathbf{I}.\end{aligned}$$

Ans (b)

8. Consider measurement of parameter h, we have each observation as

$$y(k) = hx(k) + v(k), for 1 \leq k \leq N,$$

where $v(k) \sim \mathcal{N}(0,\, \sigma^2)$, h is a gaussian parameter with $\mathrm{E}\{h\} = \mu_h$ and $\mathrm{E}\{(h-\mu_h)^2\} = \sigma_h^2$. By stacking N such observations, we obtain observation vector as

$$\mathbf{y} = \mathbf{x}h + \mathbf{v},$$

where $\mathrm{E}\{\mathbf{v}\} = \mathbf{0}$ and $\mathrm{E}\{\mathbf{v}\mathbf{v}^T\} = \sigma^2\mathbf{I}$. Similarly, $\mu_y = \mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{\mathbf{x}h + \mathbf{v}\} = \mathbf{x}\mu_h$. The observation covariance matrix is given by

$$\begin{aligned}\mathbf{R}_{yy} &= \mathrm{E}\{(\mathbf{y}-\mu_y)(\mathbf{y}-\mu_y)^T\}\\ &= \mathrm{E}\{(\mathbf{x}(h-\mu_h)+\mathbf{v})(\mathbf{x}(h-\mu_h)+\mathbf{v})^T\}\\ &= \mathrm{E}\{(h-\mu_h)^2\}\mathbf{x}\mathbf{x}^T + \mathrm{E}\{\mathbf{v}\mathbf{v}^T\} + \mathbf{x}\mathrm{E}\{(h-\mu_h)\mathbf{v}^T\} + \mathrm{E}\{\mathbf{v}(h-\mu_h)\}\mathbf{x}^T\\ &= \sigma_h^2\mathbf{x}\mathbf{x}^T + \sigma^2\mathbf{I}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{R}_{hy} &= \mathrm{E}\{(h-\mu_h)(\mathbf{y}-\mu_y)^T\}\\ &= \mathrm{E}\{(h-\mu_h)(\mathbf{x}(h-\mu_h)+\mathbf{v})^T\}\\ &= \mathrm{E}\{(h-\mu_h)^2\}\mathbf{x}^T + \mathrm{E}\{(h-\mu_h)\mathbf{v})^T\}\\ &= \sigma_h^2\mathbf{x}^T.\end{aligned}$$

The MMSE estimate of parameter h is given by

$$\hat{h} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}(\mathbf{y}-\mu_y) + \mu_h.$$

Substituting the values of the covariance matrices in the above expression, we obtain

$$\hat{h} = \sigma_h^2\mathbf{x}^T(\sigma_h^2\mathbf{x}\mathbf{x}^T + \sigma^2\mathbf{I})^{-1}(\mathbf{y}-\mathbf{x}\mu_h) + \mu_h. \quad (2)$$

Simplifying the above expression, we get

$$\begin{aligned}\sigma_h^2\mathbf{x}^T(\sigma_h^2\mathbf{x}\mathbf{x}^T+\sigma^2\mathbf{I})^{-1} &= (\sigma_h^2\mathbf{x}^T\mathbf{x}+\sigma^2)^{-1}\sigma_h^2\mathbf{x}^T\\ &= (\sigma_h^2||\mathbf{x}||^2+\sigma^2)^{-1}\sigma_h^2\mathbf{x}^T\\ &= \frac{\sigma_h^2\mathbf{x}^T}{\sigma_h^2||\mathbf{x}||^2+\sigma^2}.\end{aligned}$$

Substituting this in the equation (2), we obtain

$$\begin{aligned}\hat{h} &= \frac{\sigma_h^2\mathbf{x}^T}{(\sigma_h^2||\mathbf{x}||^2+\sigma^2)}(\mathbf{y}-\mathbf{x}\mu_h)+\mu_h\\ &= \frac{\sigma_h^2\mathbf{x}^T\mathbf{y}}{\sigma_h^2||\mathbf{x}||^2+\sigma^2}-\frac{\sigma_h^2||\mathbf{x}||^2\mu_h}{\sigma_h^2||\mathbf{x}||^2+\sigma^2}+\mu_h\\ &= \frac{\sigma_h^2\mathbf{x}^T\mathbf{y}+\sigma^2\mu_h}{\sigma_h^2||\mathbf{x}||^2+\sigma 2}\\ &= \frac{\frac{(\mathbf{x}^T\mathbf{y})/||\mathbf{x}||^2}{\sigma^2/||\mathbf{x}||^2}+\frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/||\mathbf{x}||^2}+\frac{1}{\sigma_h^2}}.\end{aligned}$$

Ans (d)

9. From the above question, the MMSE estimate is given as

$$\hat{h} = \frac{\frac{(\mathbf{x}^T\mathbf{y})/||\mathbf{x}||^2}{\sigma^2/||\mathbf{x}||^2}+\frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/||\mathbf{x}||^2}+\frac{1}{\sigma_h^2}}. \tag{3}$$

Given data: $\mu_h = 1,\ \sigma_h^2 = 1/2,\ \sigma^2 = 3\ dB \implies 10\log\sigma^2 = 3 \implies \sigma^2 = 2$

$$\mathbf{x} = \begin{bmatrix}1/2\\2\\1\\3/2\end{bmatrix}, \mathbf{y} = \begin{bmatrix}-2\\2\\-1\\1\end{bmatrix},$$

$$\begin{aligned}\mathbf{x}^T\mathbf{y} &= [1/2\ \ 2\ \ 1\ \ 3/2]\begin{bmatrix}1/2\\2\\1\\3/2\end{bmatrix}\\ &= \frac{7}{2},\\ ||\mathbf{x}||^2 &= \frac{1}{4}+4+1+\frac{9}{4} = \frac{15}{2}.\end{aligned}$$

Substituting values in (3), we obtain

$$\begin{aligned}\hat{h} &= \frac{\frac{\frac{15/2}{7/2}}{\frac{2}{15/2}} + \frac{1}{1/2}}{\frac{1}{\frac{2}{15/2}} + \frac{1}{1/2}} \\ &= \frac{15}{23}.\end{aligned}$$

Ans (a)

10. From above problem, the MMSE estimate can be written as

$$\begin{aligned}\hat{h} &= \frac{\frac{(\mathbf{x}^T\mathbf{y})/||\mathbf{x}||^2}{\sigma^2/||\mathbf{x}||^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/||\mathbf{x}||^2} + \frac{1}{\sigma_h^2}} \\ &= \frac{\frac{\text{MLE}}{\text{var of MLE}} + \frac{\text{Prior Mean}}{\text{Prior Var}}}{\frac{1}{\text{Var of MLE}} + \frac{1}{\text{Prior Var}}}.\end{aligned}$$

As $\sigma^2 \to \infty$, Var of MLE $\to \infty$ or $\sigma^2/N >> \sigma_h^2$

$$\hat{h} \to \frac{\frac{\text{Prior Mean}}{\text{Prior Var}}}{\frac{1}{\text{Prior Var}}} = \text{Prior Mean} = \mu_h = 1.$$

Ans (c)