NPTEL MOOC Estimation: Assignment #1 (Solutions)

1. In the context of the estimation framework considered, the probability density function (PDF) of the unknown parameter *h* is termed as the Prior Density (*p(h)*).

   Ans (c)

2. In the context of the estimation framework considered, the probability density function (PDF) of the unknown parameter *h* conditioned on the observation vector **y ()** is termed as the Posterior Density

   Ans (d)

3. The estimation framework described in the lectures is, Minimum Mean Squared Error Estimation.

   Ans (b)

4. Given the parameter *h* and observation vector **y**, the expression for the MMSE estimate is,

$$E\left\{\left(\hat{h}(\boldsymbol{y})-h\right)^2\right\} = \iint_{-\infty}^{\infty}\left(\hat{h}(\boldsymbol{y})-h\right)^2 p(\boldsymbol{y},h)\,dh\,d\boldsymbol{y}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\left(\hat{h}(\boldsymbol{y})-h\right)^2 p(h|\boldsymbol{y})\,dh\right\}p(\boldsymbol{y})d\boldsymbol{y}$$

MSE for a fixed value of **y**,

MMSE Estimate

$$=\arg\ min \int_{-\infty}^{\infty}\left(\hat{h}(\boldsymbol{y})-h\right)^2 p(h|\boldsymbol{y})\,dh$$

Differentiating w. r. t. $\hat{h}$ and setting to zero,

$$\frac{\partial}{\partial \hat{h}}\left[\int_{-\infty}^{\infty}\left(\hat{h}(\boldsymbol{y})-h\right)^2 p(h|\boldsymbol{y})dh\right] = 0$$

$$\int_{-\infty}^{\infty} 2\left(\hat{h}(\boldsymbol{y})-h\right)p(h|\boldsymbol{y})\,dh = 0$$

$$\hat{h}(\boldsymbol{y})\int_{-\infty}^{\infty} p(h|\boldsymbol{y})\,dh = \int_{-\infty}^{\infty} h p(h|\boldsymbol{y})\,dh$$

$$\hat{h} = E\{h|\boldsymbol{y}\}$$

   Ans (a)

5. Given ,

$$h \sim \aleph(0, \sigma_h^2),$$

$$y \sim \aleph\left(0, \sigma_y^2\right)$$

The marginal probability density functions of h, y are,

$$f_H(h) = \frac{1}{\sqrt{2\pi\sigma_h^2}} e^{\frac{-h^2}{2\sigma_h^2}},$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{\frac{-y^2}{2\sigma_y^2}}$$

Ans (d)

6. The joint probability density function of h, y is,

$$f_{H,Y}(h,y) = \frac{1}{\sqrt{(2\pi)^2|R|}} e^{-\frac{1}{2}[h \quad y]R^{-1}\begin{bmatrix}h\\y\end{bmatrix}}$$

Where, R is covariance matrix,

$$R = E\left\{[h \quad y]\begin{bmatrix}h\\y\end{bmatrix}\right\}$$

$$= E\left\{\begin{bmatrix}h^2 & hy\\yh & y^2\end{bmatrix}\right\}$$

$$= \begin{bmatrix}\sigma_h^2 & \rho\sigma_h\sigma_y\\\rho\sigma_h\sigma_y & \sigma_y^2\end{bmatrix}$$

$$|R| = \sigma_h^2\sigma_y^2(1-\rho^2)$$

$$R^{-1} = \frac{1}{\sigma_h^2\sigma_y^2(1-\rho^2)}\begin{bmatrix}\sigma_h^2 & -\rho\sigma_h\sigma_y\\-\rho\sigma_h\sigma_y & \sigma_y^2\end{bmatrix}$$

Putting values of $|R|$ and R^{-1} in $f_{H,Y}(h,y)$, we get

$$f_{H,Y}(h,y) = \frac{1}{\sqrt{(2\pi)^2(1-\rho^2)\sigma_y^2\sigma_h^2}} e^{\frac{-(\sigma_y^2h^2-2\rho\sigma_h\sigma_y hy+\sigma_h^2y^2)}{2(1-\rho^2)\sigma_y^2\sigma_h^2}}$$

Ans (b)

7. The conditional probability density function of h given y is,

$$f_{H|Y}(h|y) = \frac{f_{H,Y}(h,y)}{f_Y(y)}$$

Where,

$$f_{H,Y}(h,y) = \frac{1}{\sqrt{(2\pi)^2(1-\rho^2)\sigma_y^2\sigma_h^2}} e^{\frac{-(\sigma_y^2h^2-2\rho\sigma_h\sigma_y hy+\sigma_h^2y^2)}{2(1-\rho^2)\sigma_y^2\sigma_h^2}}$$

And

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{\frac{-y^2}{2\sigma_y^2}}$$

Hence,

$$f_{H|Y}(h|y) = \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_h^2}} e^{-\frac{\left(h-\rho\frac{\sigma_h}{\sigma_y}y\right)^2}{2(1-\rho^2)\sigma_h^2}}$$

Ans (a)

8. Given, $\mu_h = 1$, $\mu_y = 3$, $\sigma_h^2 = 4$, $\sigma_y^2 = 9$ and correlation coefficient $\rho = 0.85$ cross-correlation,

$$r_{hy} = E\{hy\}$$

We know that

$$\rho = \frac{E\{hy\}}{\sqrt{E\{h^2\}E\{y^2\}}}$$

$$= \frac{r_{hy}}{\sqrt{\sigma_y^2\sigma_h^2}}$$

$$= \frac{r_{hy}}{\sigma_h\sigma_y}$$

$$r_{hy} = \rho\sigma_h\sigma_y = (0.85)(2)(3)$$

$$r_{hy} = 5.1$$

Ans (d)

9. Given , $\mu_h = 1$, $\mu_y = 3$, $\sigma_h^2 = 4$, $\sigma_y^2 = 9$ and correlation coefficient $\rho = 0.85$. The MMSE estimate

$$\hat{h} = E\{h|\boldsymbol{y}\}$$

$$\hat{h} = r_{hy} r_{yy}^{-1} (y - \mu_y) + \mu_h$$

$$r_{hy} = 5.1$$

$$r_{yy} = \sigma_y^2 = 9$$

$$\hat{h} = \frac{5.1}{9}(y - 3) + 1$$

$$\hat{h} = 0.566y - 0.7$$

Ans: (c)

10. Given , $\mu_h = 1$, $\mu_y = 3$, $\sigma_h^2 = 4$, $\sigma_y^2 = 9$ and correlation coefficient $\rho = 0.85$
MMSE

$$r_{ee} = E\{(\hat{h} - h)^2\}$$

$$= r_{hh} - r_{hy} r_{yy}^{-1} r_{yh}$$

Where,

$$r_{yy} = \sigma_y^2 = 9$$

$$r_{hh} = \sigma_h^2 = 4$$

$$r_{hy} = \rho = 5.1$$

Hence,

$$r_{ee} = 4 - \frac{(5.1)^2}{9} = 1.11$$

Ans: (a)