Department of Physics
Indian Institute of Technology Madras

STiCM: Select/Special Topics in Classical mechanics Assignment-5

1. Prove that $\vec{\nabla}\bullet\vec{J}(\vec{r},t)=\rho \operatorname{div}\vec{v}+\vec{v}\bullet\vec{\nabla}\rho$; $\vec{J}(\vec{r},t)=$ *current density vector* $=\rho(\vec{r},t)\vec{v}(\vec{r},t)$.

2. Prove that if the velocity field of a fluid flow is in 'steady state' and solenoidal, then the fluid is 'incompressible'.

3. A mass m is placed inside a cube having each side = L. Find the gravitational flux crossing the closed surface made up of all six faces of the cube.

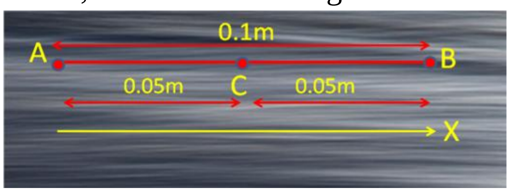
4. Prove that the acceleration due to gravity of an infinite plane of mass having uniform surface mass density is independent of the distance from the plane.

5. (a) A spherical surface surrounds a point charge *q*. What would happen to the total flux through the surface if the charge is tripled?
(b) What would happen to the total flux if the radius of the sphere is doubled?
(c) What would happen to the flux if the shape of the surface is changed from spherical to that of a regular cube of exactly the same inner volume?
(d) What would happen to the flux if the charge inside the sphere is moved from the center to another point inside the spherical surface, the new location being $(1/3)^{rd}$ the radius away from the center?

6. An insulating solid sphere of radius *R* has a uniform volume charge density ρ and it carries a total positive charge Q.
   a. Determine the magnitude of the electric field at a point $r\rangle R$.
   b. Find the magnitude of the electric field at a point $r\langle R$.
   c. Sketch $\left|\vec{E}(\left|\vec{r}\right|)\right|$ vs. $r=\left|\vec{r}\right|$ for $0\le r\le 2R$.

7. The electric field in a certain region is given by $\vec{E}(\vec{r})=kr^3\hat{e}_r$, where k is some constant.
   a. What must be the dimensions of k?
   b. Sketch the charge density $\rho(r)$ as a function of the distance from the center of the spherical polar coordinate system.
   c. Find the total charge in a sphere of radius R, centered at the origin.

8. The '*steady state*' fluid velocity *along the x-axis* shown in the figure changes from 12 m/s at point A, to 36 m/s at point B. It is also known that the velocity increases linearly with distance x along the streamlines.



Determine the acceleration of the fluid at points A, B, and C.

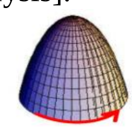
9. The *Eulerian* velocity of a fluid is given by $\vec{u}(\vec{r}) = 2z\hat{e}_x + xt\hat{e}_y + xy\hat{e}_z$ m/sec. Find the acceleration of a 'particle' of the fluid at $\vec{r}$.

10. The velocity field for a steady flow of an incompressible fluid is given by
    $v_x = -c^2 \frac{y}{\rho^2}; v_y = -c^2 \frac{x}{\rho^2}; v_z = 0$ where $\rho^2 = x^2 + y^2$ and c is a constant.
    a. Determine if the velocity field is irrotational.
    b. Determine a scalar function $\psi(x,y)$ such that $\vec{v} = -\vec{\nabla}\psi(x,y)$. If such a scalar function exists, it (i.e. the function $\psi(x,y)$) is called as the 'velocity potential'.

11. A steady irrotational flow of an incompressible, nonviscous fluid in the gravitational field is described by a velocity potential given by $\psi(x,y,z) = kyz$ where k is a constant of appropriate dimensions.
    a. Determine the velocity field.
    b. Find the difference in pressure $(P_A - P_B)$ between two points $A(y=1, z=1)$ and $B(y=2, z=2)$

12. A fire engine pumps water from a hydrant at the rate of 10^3 litres/sec. It ejects water at a nozzle $5m$ above the surface of water in the hydrant at a speed of 10 m/s.
    a. Determine the pressure difference between water at the pump and at the nozzle.
    b. Determine the power of the engine-pump. [Hint: use dimensional analysis].

13. The position vector of an arbitrary point on the rim of the flat bottom of the paraboloid surface defined by $x^2 + y^2 + z = 6$ is given by $\vec{r} = 2(\hat{e}_\rho + \hat{e}_z)$. On this surface, $z \geq 2$. We have a vector field $\vec{G} = \vec{\nabla} \times \vec{A}$ in this region, wherein $\vec{A}(\vec{r}) = z^2\hat{e}_x - 3xy\hat{e}_y + x^3y^3\hat{e}_z$.



Determine the flux $\iint \vec{G} \bullet \overrightarrow{dS}$ of $\vec{G}$ passing through the paraboloid surface. Consider the 'orientation' of the surface to be such as would be consistent with the right-hand-screw convention with reference to the bottom perimeter of the surface considered to be traversed along the red-arrow shown in the figure.

14. An azimuthal force field is given by $\vec{F}(\vec{r}) = f(r)\hat{e}_\varphi$. Find the work done by this force on a particle in moving it clockwise (as viewed from above the XY plane) along a circular path in the XY plane having a Radius R, and centered at the origin. Determine also the surface integral of the curl of the vector over the 'northern' hemispherical surface of radius R that sits on the circular path mentioned above. Is your answer consistent with the Stokes' theorem?

15. Using Stokes' Theorem to evaluate the work done by a force $\vec{F} = z^2\hat{e}_x + y^2\hat{e}_y + x\hat{e}_z$ over a closed path defined by a route described in Cartesian coordinates as point A=(0,0,2) to point B=(0,2,0) to point C=(2,0,0) and back to point A=(0,0,2). Find out, also using the Stokes' theorem, what is the work done if the path is reversed. In both cases, determine if you get the same answer by evaluating either the line integral or the surface integral that appears in the Stokes' theorem.

16. What will be your answer if the path in problem 16 was changed to B to C to A to B.

17. In a laboratory experiment, a muon was found to travel a distance of 800 m before it decayed. Typically, the muon decays in $2\times10^{-6}s$ in its own frame of reference. The speed of light is *c* m/s. Find the speed of the muon in the laboratory frame.

18. A particle has acceleration $\vec{a}$ in an inertial reference frame S at a certain instant of time. The particle's velocity in another inertial frame of reference, S', is $\vec{u}'=\vec{0}$ as $t'=0$, and its acceleration at that instant is $\vec{a}'$. S' moves relative to S along the common X-axis of both the frames at a velocity v. Obtain the transformation equations for the acceleration of the particle providing the relationship between $\vec{a}$ and $\vec{a}'$.

19. A rod of length l is at rest in a reference frame *S*. It is placed such that it makes an angle α with respect to the x-axis of a Cartesian reference frame in S. Find the length l' of the rod and the angle it would make with the x'-axis of a coordinate system S' moving with respect to *S* at a uniform speed v along their common X,X'-axis.

20. Arjun and Karan have a cup of coffee at the North pole just before Arjun sets-off on a space voyage toward the pole star in a rocket that moves relative to the earth at speed $\frac{2}{3}c$. Arjun travels for 2 years measured in his own clock. How much has Arjun aged, as per his clock, and how much has Karan aged in the corresponding duration? Arjun misses his coffee, and decides to come back, taking another 2 years to get back. Now, Arjun and Karan are not twins, they are not even brothers! So, is there any 'paradox' at all related to their relative ageing? <u>*Exactly what is it*</u>? Resolve it <u>*all the way*</u>!

21. Use your calculators to evaluate the logistic map $P_{n+1}=rP_n(1-P_n)$ for the initial condition $P_1=0.03$ and the fecundity $r=3.46$. Tabulate your results for integers $1\le n\le 20$. If you wish, you may write a small numerical program to generate the results. Sketch (*neatly!*) the graph P_n vs n and include both the table and the graph it in your submission report.

22. What is meant by Hausdorff dimension? What is a Menger sponge? Find its Hausdorff dimension.

23. Define (a) Feigenbaum constant (b) Koch curve (c) 'strange attractor' (d) Mandelbrot set.
