Week 8 - Assignment 8 - Solution

(1) (d) The angular velocity ($\vec{\omega}$) is parallel to the angular momentum ($\vec{L}$).

(2) (b) for symmetric top, $I_1 = I_2 \neq I_3$.

(3) (b) Space Z-Component of angular momentum.

(4) (a) for rotation around an axis with lowest M.O.I.

(5) (d) $I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = N_1$

(6) (a) $\dfrac{dT}{dt} = \vec{N}\cdot\vec{\omega}$

(7) (a) $\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\sin\psi$, $\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$

and $\dot{\phi}\cos\theta + \dot{\psi}$

(8) Choose the axis of symmetry coincident with one of the principal axes, say along $\hat{e}_3$, then $I_1 = I_2$

and Euler eqn become

$$I_1\dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3 = 0 \quad \text{--- (1)}$$

$$I_1\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = 0 \quad \text{--- (2)}$$

$$I_3\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{Const.} = A \text{ (say)} \quad \text{--- (3)}$$

$\therefore$ (1) and (2) become

$$\dot{\omega}_1 + \frac{I_3 - I_1}{I_1}A\omega_2 = 0 \quad \text{--- (4)} \qquad \text{and} \qquad \dot{\omega}_2 + \frac{I_1 - I_3}{I_1}A\omega_1 = 0 \quad \text{--- (5)}$$

$\therefore$ from (4) & (5)

$$\ddot{\omega}_2 + \frac{I_1 - I_3}{I_1}A\dot{\omega}_1 = 0 \quad \text{(Diff. eqn (5) w.r.t } t\text{)}$$

$$\Rightarrow \ddot{\omega}_2 + \left(\frac{I_3 - I_1}{I_1}\right)^2 A^2\omega_2 = 0$$

$$\Rightarrow \ddot{\omega}_2 + \kappa^2 \omega_2 = 0 \qquad \text{where, } K = \left| \frac{I_3 - I_1}{I_1} \right| A = 2\pi f$$

$$\therefore \omega_2 = B\cos \kappa t + C \sin \kappa t$$

$$\boxed{f = \frac{1}{2\pi} \left| \frac{I_3 - I_1}{I_1} \right| A}$$

(9) We can write K.E. $T = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right)$

$$\Rightarrow T = \frac{1}{2} I_1 \left(\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi\right)^2 + \frac{1}{2} I_2 \left(\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi\right)^2$$
$$+ \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi}\right)^2$$

[using ang. velocity in terms of Eulers angle]

For symmetric top, $I_1 = I_2 \neq I_3$

$$\therefore T = \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi}\right)^2$$

(10) For spherical bodies, $I_1 = I_2 = I_3$.

$\therefore$ From expression of T in Q.9 (above)

$$T = \frac{1}{2} I_1 \left[\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \left(\dot{\phi} \cos\theta + \dot{\psi}\right)^2 \right]$$

$$= \frac{1}{2} I_1 \left[\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \dot{\phi}^2 \cos^2\theta + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos\theta \right]$$

$$= \frac{1}{2} I_1 \left[\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos\theta \right]$$