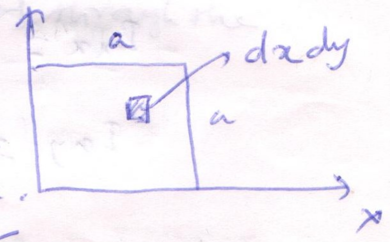
Week-7 – Assignment-7 Solution

(1)



$$I_{xx} = \int_{x=0}^{a}\int_{y=0}^{a} (\sigma\, dx\, dy)\, y^2$$

, σ = mass per unit area $= M/a^2$

$$= \sigma \int_0^a dx \int_0^a y^2\, dy$$

$$= \sigma \cdot a \cdot \frac{a^3}{3} = \frac{1}{3}\sigma a^4 = \frac{1}{3}\cdot\frac{M}{a^2}\cdot a^4$$

$$\Rightarrow \boxed{I_{xx} = \frac{1}{3} M a^2}$$

Similarly, $I_{yy} = \int_{x=0}^{a}\int_{y=0}^{a} (\sigma\, dx\, dy)\, x^2 = \frac{1}{3} M a^2$

(2)

$$I_{zz} = \int_{x=0}^{a}\int_{y=0}^{a} (\sigma\, dx\, dy)\,(x^2+y^2)$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} \sigma\, dx\, dy \cdot x^2 + \int_{x=0}^{a}\int_{y=0}^{a} \sigma\, dx\, dy \cdot y^2$$

$$= \frac{1}{3} M a^2 + \frac{1}{3} M a^2 = \frac{2}{3} M a^2$$

(3) I_{xy} = Product of inertia of the plate about x and y ~~axis~~ axes

$$= -\int_{x=0}^{a}\int_{y=0}^{a} (\sigma\, dx\, dy)\, xy = -\sigma\left(\frac{x^2}{2}\right)_0^a \left(\frac{y^2}{2}\right)_0^a$$

$$= -\sigma \cdot \frac{a^2}{2}\cdot\frac{a^2}{2} = -\frac{M}{a^2}\cdot\frac{a^4}{4} = -\frac{1}{4} M a^2$$

Since, distance of element $dxdy$ from yz, xz and xy planes are, x, y and 0 respectively.

$\therefore\ I_{xz} = I_{zx} = 0$ and $I_{yz} = I_{zy} = 0$

④ For Principal M.I.,

$$\begin{vmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-I \end{vmatrix} = 0$$

for, uniform square plate of length a,

$$I_{xx} = I_{yy} = \frac{1}{3}Ma^2, \quad I_{33} = \frac{2}{3}Ma^2$$

$$I_{xy} = I_{yx} = -\frac{1}{4}Ma^2, \quad I_{zx} = I_{xz} = I_{yz} = I_{zy} = 0$$

$$\therefore \begin{vmatrix} \frac{1}{3}Ma^2 - I & -\frac{1}{4}Ma^2 & 0 \\ -\frac{1}{4}Ma^2 & \frac{1}{3}Ma^2 - I & 0 \\ 0 & 0 & \frac{2}{3}Ma^2 - I \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{1}{3}Ma^2 - I\right)\left[\left(\frac{1}{3}Ma^2 - I\right)\left(\frac{2}{3}Ma^2 - I\right) - 0\right]$$
$$- \left(-\frac{1}{4}Ma^2\right)\left[\left(-\frac{1}{4}Ma^2\right)\left(\frac{2}{3}Ma^2 - I\right) - 0\right] = 0$$

$$\Rightarrow \left[\left(\frac{1}{3}Ma^2 - I\right)^2 \cancel{\left(\frac{1}{3}Ma^2 - I\right)} - \frac{1}{16}M^2a^4\right]\left(\frac{2}{3}Ma^2 - I\right) = 0$$

$$\Rightarrow \left[\frac{1}{9}M^2a^4 - \frac{2}{3}Ma^2 I + I^2 - \frac{1}{16}M^2a^4\right]\left(\frac{2}{3}Ma^2 - I\right) = 0$$

$$\Rightarrow \left[I^2 - \frac{2}{3}Ma^2 I + \frac{7}{144}M^2a^4\right]\left[\frac{2}{3}Ma^2 - I\right] = 0$$

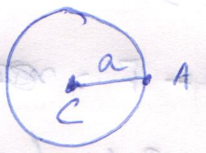
$$I^2 - \frac{2}{3}Ma^2 I + \frac{7}{144}M^2a^4 = 0$$

$$\Rightarrow I = \frac{\frac{2}{3}Ma^2 \pm \sqrt{\frac{4}{9}M^2a^4 - 4\cdot 1\cdot \frac{7}{144}M^2a^4}}{2}$$

$$= \frac{\frac{2}{3}Ma^2 \pm \sqrt{\frac{1}{4}M^2a^4}}{2} = \frac{1}{2}\left(\frac{2}{3}Ma^2 \pm \frac{1}{2}Ma^2\right)$$

$$\therefore I_1 = \frac{1}{12}Ma^2, \quad I_2 = \frac{7}{12}Ma^2 \quad \text{and} \quad I_3 = \frac{2}{3}Ma^2$$

(5) Figure shows the cross section of a circular plate [you can assume this as cross-section of a cylinder also]



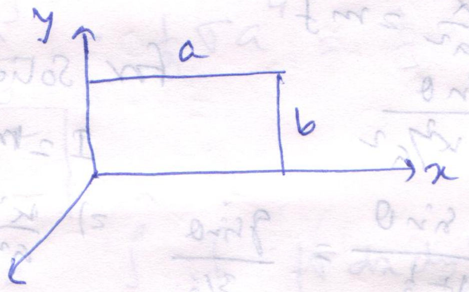
$\therefore \; I_A = I_c + Ma^2$ [using Parallel axis theorem]

We know, M.I. of a circular plate passing through the axis and C.M. $= \frac{1}{2}Ma^2$

$$\therefore \; I_A = \frac{1}{2}Ma^2 + Ma^2 = \cancel{\frac{3}{2}Ma^2}$$

$$= \frac{3}{2}Ma^2$$

(6)



$I_x = \frac{1}{3}Mb^2$

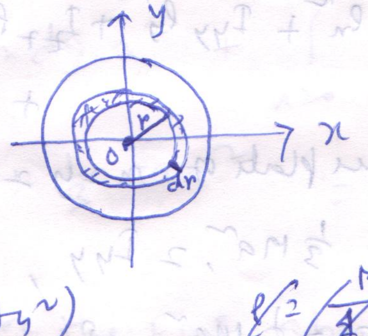
$I_y = \frac{1}{3}Ma^2$

$\therefore \; I_z = I_x + I_y$

$= \frac{1}{3}M(b^2+a^2)$ = M.I. about the vertex.

$\therefore$ Radius of Gyration $= \sqrt{I/M} = \sqrt{\frac{1}{3}(a^2+b^2)}$

(7) For spherical shell,



For spherical shell,

$\rho = \frac{M}{4\pi R^2 dR}$

$x = R\sin\theta\cos\phi$

$y = R\sin\theta\sin\phi$

$z = R\cos\theta$

$\therefore \; I_{zz} = \int \rho \, dv \, (x^2+y^2)$

$\cancel{\rho = \frac{M}{\frac{4}{3}\pi R^3}}$

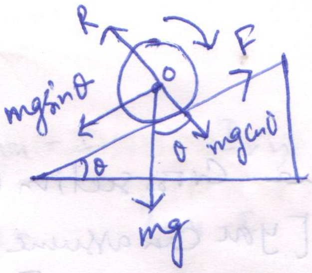
$$= \int [R^2\sin^2\theta(\underbrace{\sin^2\phi+\cos^2\phi}_{1})] \cdot \rho \cdot dR \cdot R\,d\theta \cdot r\sin\theta\,d\phi$$

$$= \rho R^4 dR \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{M}{4\pi R^2 dR} \cdot R^4 dR \cdot \frac{4}{3} \cdot 2\pi = \frac{2}{3}MR^2$$

$\therefore$ Radius of Gyration $= \sqrt{I/M} = \sqrt{\frac{2}{3}}\,R$

(8) F = frictional force
R = normal reaction



$$mg\sin\theta - F = mf$$

$$Fa = I\alpha \quad \text{(about the C.M. at O)}$$

$$\Rightarrow F = \frac{I}{a}\alpha$$

for Rolling without slipping,

$$f = \alpha a$$

$$= \frac{mk^2}{a}\cdot\frac{f}{a}$$

$$= mf\frac{k^2}{a^2}$$

$$\therefore mg\sin\theta - F = mf$$

$$\Rightarrow mg\sin\theta - mf\frac{k^2}{a^2} = mf$$

$$\Rightarrow f = \frac{g\sin\theta}{1+k^2/a^2}$$

for Solid cylinder,

$$I = mk^2 = \frac{1}{2}ma^2$$

$$\Rightarrow \frac{k^2}{a^2} = \frac{1}{2}$$

$$= \frac{g\sin\theta}{1+\frac{1}{2}} = \frac{g\sin\theta}{3/2}$$

$$\Rightarrow \boxed{f = \frac{2}{3}g\sin\theta}.$$

(9) Eqn for ellipsoid of inertia,

$$I_{xx}l_x^2 + I_{yy}l_y^2 + I_{zz}l_z^2 + 2I_{xy}l_xl_y + 2I_{yz}l_yl_z + 2I_{zx}l_zl_x = 1$$

For square plate of $x = y = a$,

$$I_{xx} = \frac{1}{3}Ma^2 = I_{yy}, \quad I_{zz} = \frac{2}{3}Ma^2$$

$$I_{xy} = -\frac{1}{4}Ma^2, \quad I_{xz} = I_{yz} = 0$$

$$\therefore \frac{1}{3}Ma^2 l_x^2 + \frac{1}{3}Ma^2 l_y^2 + \frac{2}{3}Ma^2 l_z^2 + 2\cdot\left(-\frac{1}{4}Ma^2\right)l_xl_y + 0 + 0 = 1$$

$$\Rightarrow \boxed{l_x^2 + l_y^2 + 2l_z^2 - \frac{3}{2}l_xl_y = \frac{3}{Ma^2}}$$

(10) $T = \frac{1}{2}\vec{\omega}\cdot\vec{\Omega}$ $\qquad \vec{\omega} = \hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z$

$\qquad\qquad\qquad\qquad \vec{\Omega} = \hat{i}\Omega_x + \hat{j}\Omega_y + \hat{k}\Omega_z$

$= \frac{1}{2}(\omega_x\Omega_x + \omega_y\Omega_y + \omega_z\Omega_z)$

$= \cancel{\frac{1}{2}[(\omega_x\, I_{xx}\,\omega_x}$

$= \frac{1}{2}\Big[\omega_x(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)$

$\quad + \omega_y(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)$

$\quad + \omega_z(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\Big]$

Here, $I_{xy} = I_{yx}$, $I_{zx} = I_{xz}$, $I_{yz} = I_{zy}$

$\therefore T = \frac{1}{2}\big(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2$

$\quad + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z$

$\quad + 2I_{xz}\omega_x\omega_z\big).$