Week-5 - Assignment-5 solution

① $V(r) = Kmr^3$

for circular orbit, attractive force = centrifugal force

$$\Rightarrow \frac{dV}{dr} = \frac{mv^2}{r} \Rightarrow \frac{d}{dr}(Kmr^3) = \frac{mv^2}{r}$$

$$\Rightarrow 3Kr^2 = v^2/r$$

$$\Rightarrow v\big|_{r=a} = a\sqrt{3Ka}$$

$\therefore$ Angular momentum, $L = mva$

$$\Rightarrow L = m \cdot a\sqrt{3Ka} \cdot a = \underline{\underline{\sqrt{3K}\, m a^{5/2}}}.$$

② $V' = V(r) + \frac{l^2}{2mr^2} = \frac{1}{2}Kr^2 + \frac{l^2}{2mr^2}$

for circular orbit, V' would be minimum.

$$\therefore \frac{dV'}{dr} = 0 = Kr - \frac{l^2}{mr^3}$$

$$\Rightarrow l^2 = mKr^4 \Rightarrow (mvr)^2 = mKr^4$$

$$\Rightarrow (mr^2\omega)^2 = mKr^4 \quad (\because v = \omega r)$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{K}{m}}}$$

③ $r = e^{-\theta} \Rightarrow u = \frac{1}{r} = e^{\theta}$

$$\therefore \frac{du}{d\theta} = e^{\theta} \text{ and } \frac{d^2u}{d\theta^2} = e^{\theta}$$

The eqn of orbit,

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$$

$$\Rightarrow e^{\theta} + e^{\theta} = -\frac{m}{l^2} \cdot e^{-2\theta} \cdot f\left(\frac{1}{u}\right)$$

$$\Rightarrow f\left(\frac{1}{u}\right) = -\frac{l^2}{m} \cdot 2e^{3\theta} = -\frac{l^2}{m} \cdot 2u^3$$

$$\Rightarrow f(r) = -\frac{l^2}{m} \cdot \frac{2}{r^3} \Rightarrow \boxed{f(r) \propto \frac{1}{r^3}}$$

④ $\omega' = \frac{2\pi}{T'} \Rightarrow T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\omega \sin\lambda} = \frac{T}{\sin\lambda}$

$$\Rightarrow T' = \frac{24 \text{ hr.}}{\sin 30^\circ} \Rightarrow \boxed{T' = 48 \text{ hr.}}$$

(5) $\vec{v}' = 100\ m/s\ \hat{j}'$, $\vec{\omega} = \omega\cos\lambda\,\hat{j}' + \omega\sin\lambda\,\hat{k}'$

$\lambda = 30^\circ$ $\quad = \frac{\sqrt{3}}{2}\omega\,\hat{j}' + \frac{\omega}{2}\hat{k}'$

Coriolis force $= \vec{F}_c = -2m(\vec{\omega}\times\vec{v}')$

$= -2m\left[\left(\frac{\sqrt{3}}{2}\omega\,\hat{j}' + \frac{\omega}{2}\hat{k}'\right)\times 10^2\,\hat{j}'\right]$

$= -2m\left[-\frac{\omega}{2}10^2\,\hat{i}'\right] = m\omega 10^2\,\hat{i}'$

$= 1.5\times 7.29\times10^{-5}\times 100\,\hat{i}$

$= 0.0109\,\hat{i}$ ⇒ along east.

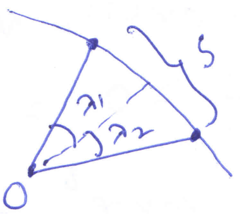
(6) $T = \frac{24}{\sin\lambda}$ hr. $\quad T_1 = \frac{24}{\sin\lambda_1}$ hr. $\Rightarrow \lambda_1 = \sin^{-1}\left(\frac{24}{T_1}\right)$

$= \sin^{-1}(24/27.6)$

$= 60.41^\circ$

Similarly, $\lambda_2 = \sin^{-1}(24/42.9)$

$= 34.02^\circ$



$s = R(\lambda_1+\lambda_2)$

$\Rightarrow R = \frac{s}{\lambda_1+\lambda_2} = \frac{10{,}466}{60.41^\circ + 34.02^\circ} = \frac{10{,}466}{94.43\times\frac{\pi}{180}} \approx \underline{6350\ km}$.

(7) $\omega = \frac{2\pi}{T} = \frac{2\pi}{23\,hr.\,56\,mins} = \frac{2\pi}{86{,}160}$ rad/s $= \underline{7.29\times10^{-5}\ rad/sec}$.

(8) $\vec{\omega} = \omega\cos\lambda\,\hat{j}' + \omega\sin\lambda\,\hat{k}' = \omega\hat{k}'$ $(\because \lambda = 90^\circ)$

$\vec{v}' = 8\ km/day\ (-\hat{i}') = \frac{8000}{86400}(-\hat{i}')$

$\vec{F}_c = -2m(\vec{\omega}\times\vec{v}') = -2m\left[\omega\hat{k}'\times\frac{8000}{86{,}400}(-\hat{i}')\right]$

$= 2m\omega\times \cancel{86400}\ \frac{8000}{86400}\,\hat{j}'$

$= 2\times5\times10^8\times7.29\times10^{-5}\times\frac{80}{864}\,\hat{j}' = 6730\ N\ \hat{j}'$
⇓
along North

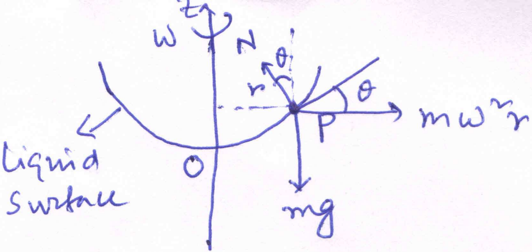
(9) $F = 2mv\omega\sin\lambda$

$= 2\times10^6\times15\times7.27\times10^{-5}\times\sin60^\circ$

$= 1889$ and on the right rail.

(10) $N\cos\theta - mg = 0$ — (1)

$-N\sin\theta + m\omega^2 r = 0$ — (2)



from (1) and (2)

$$\tan\theta = \frac{m\omega^2 r}{mg} = \frac{\omega^2 r}{g}$$

The slope of the surface at P is, $\frac{dz}{dr} = \tan\theta = \frac{\omega^2 r}{g}$

$\Rightarrow \int dz = \frac{\omega^2}{g}\int r\,dr + c \Rightarrow z = \frac{1}{2}\frac{\omega^2 r^2}{g} + c$

at, $z = 0$, $r = 0$ $\therefore$ $\boxed{z = \frac{1}{2}\frac{\omega^2 r^2}{g}}$