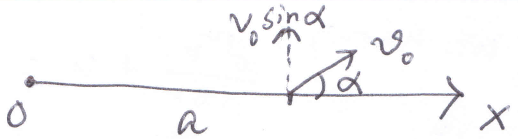
① $F(r) = -\frac{k}{r^3}$



We know,

$m(\ddot{r} - r\dot{\theta}^2) = F(r) = -\frac{k}{r^3}$

$L = |\vec{r} \times \vec{p}| = rp\sin\alpha$

$\Rightarrow mr^2\dot{\theta} = amv_0\sin\alpha$

$\Rightarrow r^4\dot{\theta}^2 = a^2v_0^2\sin^2\alpha$

$\Rightarrow r\dot{\theta}^2 = \frac{a^2v_0^2\sin^2\alpha}{r^3}$

$\Rightarrow \ddot{r} = -\frac{k}{mr^3} + \frac{a^2v_0^2\sin^2\alpha}{r^3}$

$$\Rightarrow \boxed{\frac{d^2r}{dt^2} = -\frac{k - ma^2v_0^2\sin^2\alpha}{mr^3}}$$

② let, in general, $r^n = a^n\cos n\theta$

$\Rightarrow u^n a^n \cos n\theta = 1$ ——(1)

$\Rightarrow n\ln u + n\ln a + \ln(\cos n\theta) = 0$ (Taking $\log_e$ both side)

$\Rightarrow n\cdot\frac{1}{u}\frac{du}{d\theta} + 0 - \frac{1}{\cos n\theta}\cdot\sin n\theta\cdot n = 0$

$\Rightarrow \frac{1}{u}\frac{du}{d\theta} = \tan n\theta \Rightarrow \boxed{\frac{du}{d\theta} = u\tan n\theta}$

$\Rightarrow \frac{d^2u}{d\theta^2} = \tan n\theta\frac{du}{d\theta} + u\sec^2 n\theta\cdot n = u\tan^2 n\theta + nu\sec^2 n\theta$

$\Rightarrow \frac{d^2u}{d\theta^2} + u = u(\tan^2 n\theta + 1) + nu\sec^2 n\theta = u\sec^2 n\theta + nu\sec$

$\Rightarrow F(\frac{1}{u})\cdot\frac{m}{L^2u^2} = u\sec^2 n\theta\,[n+1]$

from (1)

$u^na^n\cos n\theta = 1$

$\Rightarrow \sec n\theta = u^na^n$

$\Rightarrow F(\frac{1}{u}) = \frac{L^2u^2\cdot u}{m}\, u^{2n} a^{2n}$

$= \frac{L^2a^{2n}}{m}\, u^{2n+3}$

$$\Rightarrow \boxed{F(r) \propto \frac{1}{r^{2n+3}}}$$

when $r = a\cos\theta$

i.e. $n = 1$

$F(r) \propto \frac{1}{r^5}$

⑥ $r^2 = a^2\cos 2\theta$

Here $n = 2$ $\therefore F(r) \propto \frac{1}{r^7}$

3 $r = ae^{\theta}$ ——(1)

$\Rightarrow \dot{r} = ae^{\theta}\dot{\theta} = r\dot{\theta}$ ——(2)

$\Rightarrow \ddot{r} = ae^{\theta}\ddot{\theta} + ae^{\theta}\dot{\theta}^2 = r(\ddot{\theta}+\dot{\theta}^2]$

$\Rightarrow \ddot{r} - r\dot{\theta}^2 = r\ddot{\theta}$ = Radial comp. of acceleration

$= a_r = 0$

$\therefore r\ddot{\theta} = 0$

$\Rightarrow \boxed{\dot{\theta} = \text{Const} = \omega}$ $[r \neq 0]$

$\Downarrow$

Angular velocity is constant.

Magnitude of velocity $= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$

$\Rightarrow |v| = \sqrt{r^2\dot{\theta}^2 + r^2\dot{\theta}^2} = \sqrt{2}\, r\dot{\theta}$

$\Rightarrow \boxed{|v| \propto r}$ [as $\dot{\theta}$ = const]

(4) We know, $\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2}$ $\quad$ $F(r) = -k/r^2$

$\Rightarrow F(\frac{1}{u}) = -ku^2$

$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2}$

It has general solⁿ, $u = A\cos\theta + B\sin\theta + k/mh^2$

$\Rightarrow u = \frac{k}{mh^2} + C\cos(\theta - \phi)$

It is always possible to choose an axis for that $\phi = 0$

$\therefore u = \frac{k}{mh^2} + C\cos\theta \quad$ —(1)

Equⁿ of conic section, $\frac{p}{r} = 1 + \epsilon\cos\theta \Rightarrow u = \frac{1}{p} + \frac{\epsilon}{p}\cos\theta$ —(2)

from (1) & (2),

$p = mh^2/k$ and $\epsilon = \frac{mh^2C}{k}$

Again, we know, $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{mh^2}$

$V = -\int f(r)\,dr = \int k/r^2\,dr = -\frac{k}{r} = -ku$

$\Rightarrow (C\sin\theta)^2 + \left(\frac{k}{mh^2} + C\cos\theta\right)^2 = \frac{2(E+ku)}{mh^2}$

$\Rightarrow C^2 + \frac{k^2}{m^2h^4} + \frac{2kC\cos\theta}{mh^2} = \frac{2E}{mh^2} + \frac{2k}{mh^2}\left[\frac{k}{mh^2} + C\cos\theta\right]$ [using (1)]

$\Rightarrow C^2 = \frac{2E}{mh^2} + \frac{k^2}{m^2h^4} = \frac{k^2}{m^2h^4}\left[\frac{2Emh^2}{k^2} + 1\right]$

for hyperbola,

$p = a(\epsilon^2 - 1) = a\left[\frac{m^2h^4C^2}{k^2} - 1\right]$

$\Rightarrow \frac{mh^2}{k} = a \cdot \frac{2Emh^2}{k^2}$

$\Rightarrow E = \frac{k}{2a}$

$\therefore \frac{1}{2}mv^2 = E - V = \frac{k}{2a} - \left(-\frac{k}{r}\right)$

$\Rightarrow v^2 = \frac{2}{m}\left(\frac{k}{2a} + \frac{k}{r}\right) \Rightarrow \boxed{v^2 = \frac{k}{m}\left(\frac{1}{a} + \frac{2}{r}\right)}$

(5) $\frac{mv^2}{R} = \frac{GMm}{R^2}$

$\Rightarrow v = \sqrt{\frac{GM}{R}}$

$T = \frac{2\pi R}{v} = 2\pi R\sqrt{\frac{R}{GM}} = 2\pi\sqrt{\frac{R^3}{GM}}$

$M = \frac{4}{3}\pi R^3 \rho$

$\therefore T = 2\pi\sqrt{\frac{\cancel{R^3}}{G\times\frac{4}{3}\pi \cancel{R^3}\rho}}$

$\Rightarrow \boxed{T = 2\pi\sqrt{\frac{3}{4\pi G\rho}}}$

Yes, period is independent of R of the planet.

7. $T^2 = 4\pi^2 m a^3 / k$ (Kepler's 3rd law)

$\Rightarrow \boxed{T = 2\pi a^{3/2} \sqrt{m/k}}$

(8) $f(r) = \frac{mh^2}{r^4}\left[\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r\right]$

$r = a(1-\cos\theta)$

$\frac{dr}{d\theta} = a\sin\theta$

$\frac{d^2r}{d\theta^2} = a\cos\theta$

$= \frac{mh^2}{r^5}\left[r\frac{d^2r}{d\theta^2} - 2\left(\frac{dr}{d\theta}\right)^2 - r^2\right]$

$= \frac{mh^2}{r^5}\left[a(1-\cos\theta)\cdot a\cos\theta - 2\cdot a^2\sin^2\theta - a^2(1-\cos\theta)^2\right]$

$= \frac{mh^2}{r^5}\left[a^2\cos\theta - a^2\cos^2\theta - 2a^2\sin^2\theta - a^2(1-2\cos\theta+\cos^2\theta)\right]$

$= \frac{mh^2a^2}{r^5}\left[\cos\theta - \underline{\cos^2\theta} - 2\sin^2\theta - 1 + 2\cos\theta - \underline{\cos^2\theta}\right]$

$= \frac{mh^2a^2}{r^5}\left[3\cos\theta - 2(\sin^2\theta + \cos^2\theta) - 1\right]$

$= \frac{mh^2a^2}{r^5}\left[3\cos\theta - 3\right]$

$= \frac{3mh^2a}{r^5}\cdot a(\cos\theta - 1) = \frac{3mh^2a}{r^5}\cdot(-r) = -\frac{3mh^2a}{r^4}$

$\Rightarrow \boxed{f(r) \propto r^{-4}}$

(9) Areal velocity $= \frac{1}{2} r^2 \dot{\theta}$

(10) On the surface of the earth,

$$\frac{GMm}{R^2} = mg \Rightarrow \boxed{g = \frac{GM}{R^2}}$$