Week 12 - Assignment 12 Solution

① (b) have all elements real and positive.

② $V(x) = x^4 - 4x^3 - 8x^2 + 48x$

For any equ^m point, $\frac{dV}{dx} = 0$

$\Rightarrow 4x^3 - 12x^2 - 16x + 48 = 0$

$\Rightarrow x^3 - 3x^2 - 4x + 12 = 0$

$\Rightarrow x^2(x-3) - 4(x-3) = 0$

$\Rightarrow (x-3)(x^2-4) = 0 \Rightarrow x = 3, \pm 2$

$\frac{d^2V}{dx^2} = 12x^2 - 24x - 16$

$\left.\frac{d^2V}{dx^2}\right|_{x=2} = 48 - 48 - 16 = -16 < 0 \Rightarrow$ unstable equ^m

$\left.\frac{d^2V}{dx^2}\right|_{x=-2} = 48 + 48 - 16 = 80 > 0 \Rightarrow$ stable equ^m

$\left.\frac{d^2V}{dx^2}\right|_{x=3} = 108 - 72 - 16 = 20 > 0 \Rightarrow$ Stable equ^m

$\therefore$ Stable at $x = -2, 3$.
Unstable at $x = 2$.

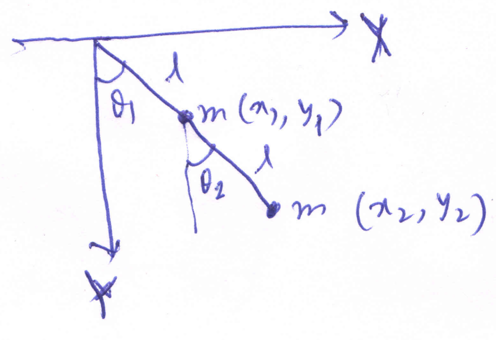
③ (d) Fig. d does not represent normal modes.
As in this case, the molecules is translating as a whole downward, its centre of mass is not fixed.

④ $x_1 = l \sin\theta_1$

$x_2 = l\sin\theta_1 + l\sin\theta_2$

$y_1 = l(1 - \cos\theta_1)$

$y_2 = l(1 - \cos\theta_1) + l(1 - \cos\theta_2)$



$\dot{x}_1 = l\cos\theta_1\,\dot{\theta}_1 , \quad \dot{x}_2 = l(\cos\theta_1\,\dot{\theta}_1 + \cos\theta_2\,\dot{\theta}_2)$

$\dot{y}_1 = l\sin\theta_1\,\dot{\theta}_1 , \quad \dot{y}_2 = l(\sin\theta_1\,\dot{\theta}_1 + \sin\theta_2\,\dot{\theta}_2)$

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}ml^2\Big[\dot{\theta}_1^2(\cos^2\theta_1 + \sin^2\theta_1) + \dot{\theta}_1^2(\cos^2\theta_1 + \sin^2\theta_1)$$
$$+ \dot{\theta}_2^2(\cos^2\theta_2 + \sin^2\theta_2) + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\Big]$$

$$= \frac{1}{2}ml^2(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2)$$ [for small angle appox. $\cos(\theta_1-\theta_2) \approx 1$]

$$2T = \begin{pmatrix}\dot{\theta}_1 & \dot{\theta}_2\end{pmatrix}\begin{pmatrix}2ml^2 & ml^2 \\ ml^2 & ml^2\end{pmatrix}\begin{pmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{pmatrix}$$

$$V = mg(y_1 + y_2) = mgl\big[(1-\cos\theta_1) + (1-\cos\theta_1) + (1-\cos\theta_2)\big]$$

$$= mgl\big[(1-1+\theta_1^2/2) + (1-1+\tfrac{\theta_1^2}{2}) + (1-1+\theta_2^2/2)\big]$$

$$2V = mgl(2\theta_1^2 + \theta_2^2)$$

$$= \begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}\begin{pmatrix}2mgl & 0 \\ 0 & mgl\end{pmatrix}$$

For non-trivial sol^n of normal modes,

$$|V - \omega^2 T| = 0$$

$$\Rightarrow \begin{vmatrix} 2mgl - 2m\omega^2 l^2 & -\omega^2 ml^2 \\ -\omega^2 ml^2 & mgl - \omega^2 ml^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2g - 2l\omega^2 & -\omega^2 l \\ -\omega^2 l & g - \omega^2 l \end{vmatrix} = 0$$

$$\Rightarrow 2(g - l\omega^2)^2 - \omega^4 l^2 = 0$$

$$\Rightarrow \quad 2g^2 - 4gl\omega^2 + 2\omega^4 l^2 - \omega^4 l^2 = 0$$

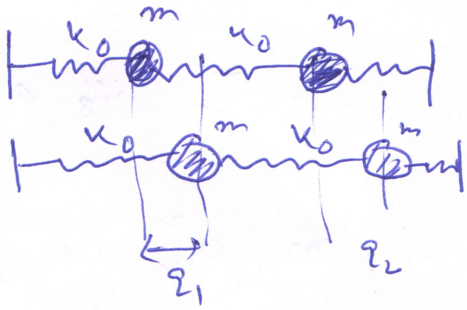
$$\Rightarrow \quad \omega^4 l^2 - 4gl\omega^2 + 2g^2 = 0$$

$$\Rightarrow \quad \omega^2 = \frac{4gl \pm \sqrt{(4gl)^2 - 4l^2 \cdot 2g^2}}{2l^2}$$

$$= \frac{4gl \pm 2gl\sqrt{4-2}}{2l^2} = (2 \pm \sqrt{2})\, g/l$$

$$\Rightarrow \quad \boxed{\omega = \sqrt{(2 \pm \sqrt{2})\, g/l}}$$

5



$$T = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2$$

$$\Rightarrow \quad 2T = (\dot{q}_1 \;\; \dot{q}_2) \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$V = \frac{1}{2} k_0 q_1^2 + \frac{1}{2} k_0 q_2^2 + \frac{1}{2} k_0 (q_2 - q_1)^2$$

$$\Rightarrow \quad 2V = k_0 (2q_1^2 + 2q_2^2 + 2q_1 q_2)$$

$$= (q_1 \;\; q_2) \begin{pmatrix} 2k_0 & -k_0 \\ -k_0 & 2k_0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$|V - \omega^2 T| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2k_0 - \omega^2 m & -k_0 \\ -k_0 & 2k_0 - m\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (2k_0 - \omega^2 m)^2 - k_0^2 = 0$$

$$\Rightarrow \quad 2k_0 - \omega^2 m = \pm k_0$$

$$\Rightarrow \quad \omega^2 = \frac{2k_0 \mp k_0}{m}$$

$$\therefore \quad \omega = \sqrt{\frac{k_0}{m}}, \; \sqrt{\frac{3k_0}{m}}$$

(6) Normal modes

$$\sum_j (V_{ij} - \omega_k^2 T_{ij}) a_{jk} = 0 \quad , \quad i=1,2.$$

$$\begin{vmatrix} 2k_0 - m\omega_k^2 & -k_0 \\ -k_0 & 2k_0 - m\omega_k^2 \end{vmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \end{pmatrix} = 0$$

for $\omega_k = \omega_1 = \cancel{\sqrt{k_0/m}}\ \sqrt{k_0/m}$

$k_0 a_{11} - k_0 a_{21} = 0$ $\Rightarrow$ $a_{11} = a_{21}$

$-k_0 a_{11} + k_0 a_{21} = 0$

Normalization Condition, $\sum_{ij} T_{ij} a_{il} a_{jk} = \delta_{lk}$

$\therefore \sum_i T_{ii} a_{ik}^2 = 1$ as $T_{ij} = 0$ for $i \neq j$

$\delta_{lk} = 0$ for $l \neq k$

$\therefore m(a_{1k}^2 + a_{2k}^2) = 1$

$\Rightarrow m(a_{11}^2 + a_{21}^2) = 1$ $\therefore a_{11} = a_{21} = \frac{1}{\sqrt{2m}}$

$\therefore a_1 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $\omega_k = \omega_2 = \sqrt{3k_0/m}$

$-k_0 a_{12} - k_0 a_{22} = 0$ $\therefore a_{12} = -a_{22}$

$-k_0 a_{12} - k_0 a_{22} = 0$

similarly, $a_2 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(7)

$$K.E.\ T = \frac{1}{2} m \dot{Q}_1^2 + \frac{1}{2} m \dot{Q}_2^2$$

$$\Rightarrow 2T = (\dot{Q}_1 \ \dot{Q}_2) \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \end{pmatrix}$$

$$P.E.\ V = \frac{1}{2} k Q_1^2 + \frac{1}{2} k Q_2^2 + \frac{1}{2} \alpha Q_2^2$$

$$2V = k Q_1^2 + (k+\alpha) Q_2^2$$

$$= (Q_1 \ Q_2) \begin{pmatrix} k & 0 \\ 0 & k+\alpha \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

$$|V - \omega^2 T| = 0$$

$$\Rightarrow \begin{vmatrix} k - m\omega^2 & 0 \\ 0 & (k+\alpha) - \omega^2 m \end{vmatrix} = 0$$

$$\Rightarrow (k - m\omega^2)(k + \alpha - m\omega^2) = 0$$

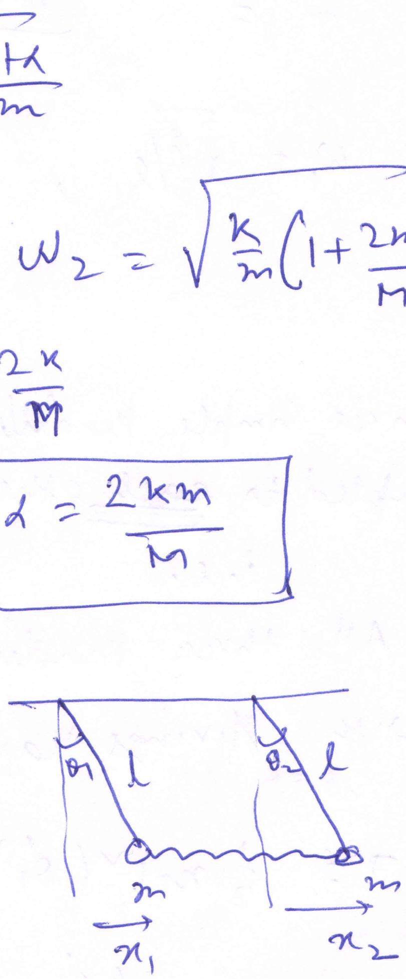
$$\therefore \omega_1 = \sqrt{k/m}, \quad \omega_2 = \sqrt{\frac{k+\alpha}{m}}$$

(8) Comparing with class note $\omega_2 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$

$$\therefore \frac{k+\alpha}{m} = \frac{k}{m} + \frac{2k}{M}$$

$$\Rightarrow \frac{\alpha}{m} = \frac{2k}{M} \Rightarrow \boxed{\alpha = \frac{2km}{M}}$$

(9) $$T = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$



$$2T = (\dot{\theta}_1 \ \dot{\theta}_2) \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$V = \frac{1}{2} mgl(\theta_1^2+\theta_2^2) + \frac{1}{2} kl^2(\theta_1^2+\theta_2^2-2\theta_1\theta_2)$$

$$\Rightarrow 2V = (mgl + kl^2)(\theta_1^2+\theta_2^2) \mp 2kl^2\theta_1\theta_2$$

$$= \begin{pmatrix}\theta_1 & \theta_2\end{pmatrix} \begin{pmatrix} mgl+kl^2 & -kl^2 \\ -kl^2 & mgl+kl^2 \end{pmatrix} \begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}$$

$$|V-\omega^2 T| = 0$$

$$\begin{vmatrix} mgl+kl^2-\omega^2 ml^2 & -kl^2 \\ -kl^2 & mgl-kl^2-\omega^2 ml^2 \end{vmatrix} = 0$$

$$\Rightarrow (mgl+kl^2-\omega^2 ml^2)^2 - k^2 l^4 = 0$$

$$\Rightarrow mgl+kl^2-\omega^2 ml^2 = \pm kl^2$$

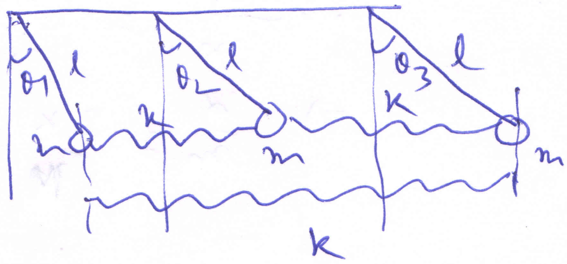
$$\Rightarrow \omega^2 = \frac{mgl+kl^2 \mp kl^2}{ml^2}$$

$$\therefore \omega_1 = \sqrt{g/l}, \quad \omega_2 = \sqrt{\frac{g}{l}+\frac{2k}{m}}$$

(10)

Three simple pendulum coupled to <u>each</u> other

i.e.



All three pendulum is connected to each other with springs having & spring constant k.

$$T = \frac{1}{2} ml^2(\dot\theta_1^2+\dot\theta_2^2+\dot\theta_3^2)$$

$$\Rightarrow 2T = \begin{pmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{pmatrix} \begin{pmatrix} ml^2 & 0 & 0 \\ 0 & ml^2 & 0 \\ 0 & 0 & ml^2 \end{pmatrix} \begin{pmatrix}\dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3\end{pmatrix}$$

$$V = \frac{1}{2} mgl(\theta_1^2+\theta_2^2+\theta_3^2) + \frac{1}{2} kl^2\left[(\theta_1-\theta_2)^2 + (\theta_2-\theta_3)^2 + (\theta_1-\theta_3^2)\right]$$

$$2V = \cancel{\tfrac{1}{2}}(mgl + kl^2)(\theta_1^2+\theta_2^2+\theta_3^2) - 2kl^2(\theta_1\theta_2+\theta_2\theta_3 + \theta_1\theta_3)$$

$$= (\theta_1 \quad \theta_2 \quad \theta_3)\begin{pmatrix} mgl+kl^2 & -kl^2 & -kl^2 \\ -kl^2 & mgl+kl^2 & -kl^2 \\ -kl^2 & -kl^2 & mgl+kl^2 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

$$|V - \omega^2 T| = 0$$

$$\Rightarrow \begin{vmatrix} mgl+kl^2-ml^2\omega^2 & -kl^2 & -kl^2 \\ -kl^2 & mgl+kl^2-ml^2\omega^2 & -kl^2 \\ -kl^2 & -kl^2 & mgl+kl^2-ml^2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (mgl+kl^2-ml^2\omega^2)\left[(mgl+kl^2-ml^2\omega^2)^2 - k^2l^4\right]$$
$$+kl^2\left[-kl^2(mgl+kl^2-ml^2\omega^2) - k^2l^4\right]$$
$$-kl^2\left[k^2l^4 + kl^2(mgl+kl^2-ml^2\omega^2)\right] = 0$$

$$\Rightarrow (mgl+kl^2-ml^2\omega^2)^3 - 3k^2l^4(mgl+kl^2-ml^2\omega^2) - 2k^3l^6 = 0$$

$$x^3 - 3k^2l^4x - 2k^3l^6 = 0 \qquad \text{where, } x = (mgl+kl^2-ml^2\omega^2)$$

$$\Rightarrow x^3 - k^2xl^4 - 2k^2xl^4 - 2k^3l^6 = 0$$

$$\Rightarrow x(x+kl^2)(x-kl^2) - 2k^2l^4(x+kl^2) = 0$$

$$\Rightarrow (x+kl^2)\left[x(x-kl^2) - 2k^2l^4\right] = 0$$

$$\Rightarrow (x+kl^2)\left[x^2 - kxl^2 - 2k^2l^4\right] = 0$$

$\Rightarrow (x+kl^2)\left[(x^2-l^4k^2) - kxl^2 - k^2l^4\right] = 0$

$\Rightarrow (x+kl^2)\left[(x+kl^2)(x-kl^2) - kl^2(x+kl^2)\right] = 0$

$\Rightarrow (x+kl^2)(x+kl^2)(x-2kl^2) = 0$

$\therefore \quad x = -kl^2 \quad \Big| \quad x = -kl^2 \quad \Big| \quad x = 2kl^2$

$\Rightarrow mgl + kl^2 - ml^2\omega^2 = -kl^2$

$\Rightarrow \omega^2 = \dfrac{mgl + kl^2 + kl^2}{ml^2} = \dfrac{mgl + 2kl^2}{ml^2} \quad \Rightarrow \omega_1 = \sqrt{\dfrac{g}{l} + \dfrac{2k}{m}}$

$\Rightarrow \cancel{\omega_1 = \sqrt{\dfrac{g}{m} + \dfrac{2k}{m}}}$

~~Similarly,~~ $\cancel{\omega_2 =}$

Similarly, $\omega_2 = \sqrt{\dfrac{g}{l} + \dfrac{2k}{m}}$

$$\omega_3 = \sqrt{\dfrac{mgl + kl^2 - 2kl^2}{ml^2}}$$

$$= \sqrt{\dfrac{g}{l} - \dfrac{k}{m}}$$