Week-10 - Assignment 10 Solution

① For dissipative function, Lagranges eqn becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = F_k = -\frac{\partial \mathcal{F}}{\partial \dot{q}_k} \quad \text{--- (1)}$$

Here, $\mathcal{F} = \frac{1}{2} k v^2 = \frac{1}{2} k \dot{y}^2$

$$\frac{\partial \mathcal{F}}{\partial \dot{y}} = k\dot{y} = kv$$

For a particle of mass m, falling under gravity from a height y,

$$K.E. = \frac{1}{2} m \dot{y}^2$$

$$P.E = -mgy$$

$$L = T - V = \frac{1}{2} m \dot{y}^2 + mgy$$

$\therefore$ from (1),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} + \frac{\partial \mathcal{F}}{\partial \dot{y}} = 0$$

$$\Rightarrow m\ddot{y} - mg + kv = 0$$

$$\Rightarrow \ddot{y} = \frac{mg - kv}{m}$$

For max. possible velocity of fall,

$$mg - kv_m = 0 \Rightarrow \boxed{v_m = mg/k}$$

② (b) $L = L(q_k, \dot{q}_k, t)$

③ In Plane Polar Co-ordinate (r, θ),

$$K.E, \; T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

Lagranges eqns are,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

$$\Rightarrow \frac{d}{dt}(m\dot{r}) - mr\dot{\phi}^2 = -kr\cos\phi$$

$$\Rightarrow \boxed{m\ddot{r} - mr\dot{\phi}^2 + kr\cos\phi = 0}$$

And, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = Q_\theta = 0 \quad (F_\theta = 0)$

$\Rightarrow \frac{d}{dt}\left(m r^2 \dot\theta\right) = 0$

$\Rightarrow \boxed{m r^2 \dot\theta = \text{const}}$

(4) Since it is moving along x-axis with uniform velocity v_0; there are no y-comp. of velocity.

$\dot x_1 = v_0 \quad , \quad x_2 = x_1 + l \sin\theta$

$\Rightarrow \dot x_2 = \dot x_1 + l \cos\theta \, \dot\theta$

$= v_0 + l \cos\theta \, \dot\theta$

$\therefore \boxed{L = \frac{1}{2} m_1 v_0^2 + \frac{1}{2} m_2 (v_0 + l \cos\theta \, \dot\theta)^2 + mgl\cos\theta}$

(5) K.E. of rotation of dipole $= \frac{1}{2} I \dot\theta^2$

M.I. of the dipole about an axis passing through its C.G. and $\perp^r$ to its length is

$I = m_1 (l/2)^2 + m_2 (l/2)^2 = \frac{l^2}{4}(m_1 + m_2)$

P.E. $= -pE\cos\theta$

$\therefore \boxed{L = \frac{l^2}{8}(m_1 + m_2)\dot\theta^2 + pE\cos\theta}$

(6) $T = \frac{1}{2} L i^2 = \frac{1}{2} L \dot q^2 \quad \left[\text{as } \frac{dq}{dt} = i\right]$

$V = \frac{1}{2} q^2/c$

$\therefore L = \frac{1}{2} L \dot q^2 - \frac{1}{2} q^2/c$

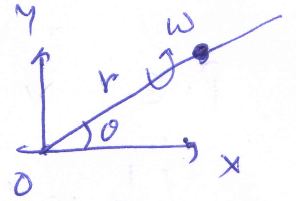
$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0$

$\Rightarrow \frac{d}{dt}(L \dot q) + q/c = 0$

$\Rightarrow \boxed{\ddot q + \frac{q}{Lc} = 0}$

(7) $L = \frac{1}{2} I \dot{\theta}^2 - mgh\cos\theta$

(8)



$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$x = r\cos\theta$
$y = r\sin\theta$
$\dot{\theta} = w$

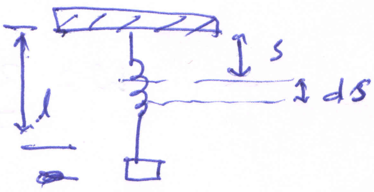
In a force free space K.E (T) is conserved.

$$\therefore T = \frac{1}{2} m (\dot{r}^2 + r^2 w^2)$$

$$\Rightarrow \dot{r}^2 = \frac{2T}{m} - r^2 w^2$$

$$\Rightarrow \boxed{\dot{r} = \sqrt{\text{Const} - w^2 r^2}}$$

(9) Let at lower end where mass m is connected, velocity is $\dot{y}$ at $s = l$, velocity is ~~maximum~~ maximum and at $s = 0$, velocity is zero.



$\therefore$ At any ~~dinstant~~ distance, s, velocity is $\left(\frac{s}{l}\right)\dot{y}$

$\therefore$ ds length of the spring, K.E.

$$dT = \frac{1}{2} \Delta m \left(\frac{s}{l}\dot{y}\right)^2 = \frac{1}{2} \rho\, ds \left(\frac{s}{l}\dot{y}\right)^2, \quad \rho = \text{mass per unit length}$$

$$T = \frac{1}{2} \int_0^l \rho\, ds \frac{s^2}{l^2} \dot{y}^2$$

$$= \frac{1}{6} \frac{\rho \dot{y}^2}{l^2} \cdot l^3 = \frac{1}{6} l \rho \dot{y}^2$$

Total mass of spring $M = \rho l$

$$\therefore T = \frac{1}{6} M \dot{y}^2$$

Total K.E. $= \left(\frac{1}{2} m \dot{y}^2 + \frac{1}{6} M \dot{y}^2\right)$

(10) From Q9, $K.E. = \frac{1}{6}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2$

P.E. $V = \frac{1}{2}ky^2$

$$L = \left(\frac{1}{6}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2\right) - \frac{1}{2}ky^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\Rightarrow \frac{1}{3}M\ddot{y} + m\ddot{y} + ky = 0$$

$$\Rightarrow \left(m + \frac{M}{3}\right)\ddot{y} + ky = 0$$

$$\Rightarrow \ddot{y} + \frac{k}{m + M/3}\,y = 0$$

$$\therefore \omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m + M/3}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{M/3 + m}{k}}$$