NPTEL course offered by IIT Madras

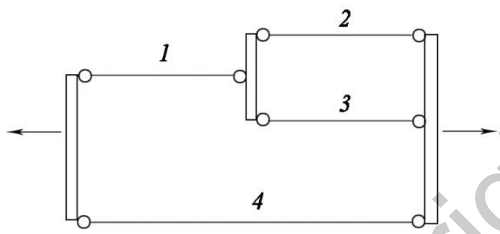
Risk and Reliability of Offshore structures

Tutorial 6: System Reliability

Answer all questions **Total marks: 25**

1. Explain General systems, in the context of system reliability

Of most interest in the field of system reliability are general systems, which can be represented schematically as shown in Figure below.



A general system is said to have failed if there is no way in which a force (information or other quantity of interest) can be transmitted from one end of the system to the other. It should be evident that series and parallel systems are special cases of a general system. To facilitate the analysis of general systems, it is useful to introduce the concepts of cut sets and path sets.

Cut sets

A cut set is any set of components whose joint failure constitutes a failure of the system. For the general system, cut sets are:

C_1={1,2,3,4} C_2={1,2,4} C_3={1,3,4} C_4={2,3,4} C_5={1,4}

A *minimum cut set* is a cut set that, upon removal of each of its components (one at a time) ceases to be a cut set. Thus, for the above example, the minimum cut sets are C_4 and C_5. Note

That C_1 is not a minimum cut set because we can remove component 1 and still have a cut set (C_4). Similarly, removing component 2 from cut set C_2 or component 3 from cut set C_3 do not cause either of these sets to cease to be a cut set.

Disjoint cut sets are cut sets that do not contain common components. For the above example there are no disjoint cut sets because each cut set contains component 4.

Path sets

An alternative to the cut set formulation of a general system is the path set formulation. A path set is any set of components whose joint survival constitutes survival of the system. For the general system considered above, the path sets are:

$P_1 = \{4\}$ $P_2 = \{1,2\}$ $P_3 = \{1,3\}$ $P_4 = \{1,4\}$ $P_5 = \{2,4\}$

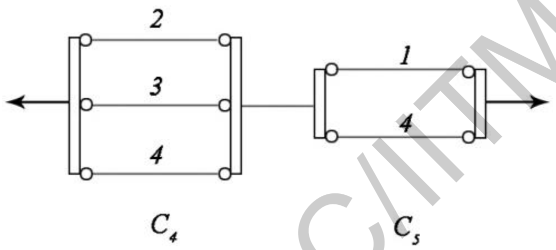
$P_6 = \{3,4\}$ $P_7 = \{1,2,3\}$ $P_8 = \{1,2,4\}$ $P_9 = \{1,3,4\}$ $P_{10} = \{2,3,4\}$ $P_{11} = \{1,2,3,4\}$

A minimum path set is a path set that, upon the removal of each of its components (one at a time) ceases to be a path set. Therefore, For the above example, the minimum path sets are P_1, P_2 and P_3. Note that for the remaining path sets. There is at least one component that can be removed without causing the set to cease to be a path set.

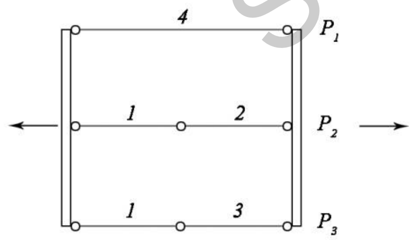
Disjoint path sets are path sets that do not contain common components. For the above example P_1 and P_2 are disjoint path sets. It is left to the reader to identify the four other pairs of disjoint path sets for this system

2. Derive system function for a general system to obtain its reliability

To define the system, it is useful to think of the system as either a series system of its minimum cut sets or as a parallel system of its minimum path sets. These representations of the general system indicated in the above question are shown detailed below:



Series system of minimum cut sets



Parallel system of minimum path sets

As shown in the above figures, each minimum cut set can be thought of as a subsystem of parallel components. Alternatively, each minimum path set can be thought of as a subsystem of components in series. For the cut set formulation, the system fails if components 2, 3 and 4 fail simultaneously and/or components 1 and 4 fail simultaneously. No other combinations of component failures constitute a failure of the system. For the path set formulation, it is easy to verify that the system survives if any of the series subsystems shown in the above figure survive. Note that inadvertently including a cut set that is not a minimum cut set in the series system shown in Figure 4 will not change the system performance. Similarly, including a path set that is not a minimum path set in the parallel system will not change the assessment of the system performance. However, in both cases, the information provided by these additional cut sets or path sets is redundant.

Let us now define the system function for the general system considered above using the cut set and path set formulations. In general, for a series system of cut sets, the system function is given by:

$a_Z = \Pi_{m-1}^{n_c} \alpha_m(a)$

Where $n_c$the number of cut is sets and α_m (a) is the cut set function of the m^{th} cut set. Recalling that when system functions for a parallel system, cut set function is given by:

$\alpha_m(a) = 1 - \Pi_{i=c_m}(1 - a_i)$

Where, C_m represents the m^{th} cut set .Thus, for the general system considered here, the system function is given by:

$a_z = [1 - (1 - a_2)(1 - a_3)(1 - a_4)]\left[\left(1 - (1 - a_1)\right)\left(1 - (a_4)\right)\right]$

Expanding the above equation, we get:

$a_z = a_4 + a_1 a_2 + a_1 a_3 - a_1 a_2 a_3 - a_1 a_2 a_4 - a_1 a_2 a_3 + a_1 a_2 a_3 a_4$

Note that in computing the above, following property is used:

$a_i^K = a_{i,} K > 0$

For Boolean variables, path set formulation will yield the system function as:

$a_z = 1 - \Pi_{i=1}^{n_e}[1 - \beta_1(a)]$

Where $n_p$the number of path is sets and β_1 (a) is the path set function of the l^{th} path set. The path set function is given by:

$\beta_1(a) = \Pi_{i \in p_i} a_i$

Where, P_1 is the $l^{\,th}$path set. Thus, for the general system considered here the system function is given by:

$a_Z = 1 - [1 - a_4]|1 - a_1 a_2|1 - a_1 a_3$

Expanding, we get:

$a_z = a_4 + a_1 a_2 + a_1 a_3 - a_1 a_2 a_3 - a_1 a_2 a_3 - a_1 a_2 a_4 - a_1 a_3 a_4 + a_1 a_2 a_3 a_4$

3. Explain how failure domains of a general system are identified to compute system reliability

Indicator functions, a_{ij}, i =1,2,….., n , for the components and a_z for the system are Bernoulli variables. Let probability of failure of i^{th} component and probability of system failure are given by:

$$P_i = P(a_j = 0)$$

$$P_F = P(a_Z = 0)$$

For the i^{th} component, following relationship holds good:

$$E[a_i] = 1(1-p_i) + 0(p_i) = 1 - p_i$$

Therefore,

$$P_i = E[\bar{a}_j]$$

Where $\bar{a}_i = 1 - a_i$ is the complementary event to a_i. Similarly,

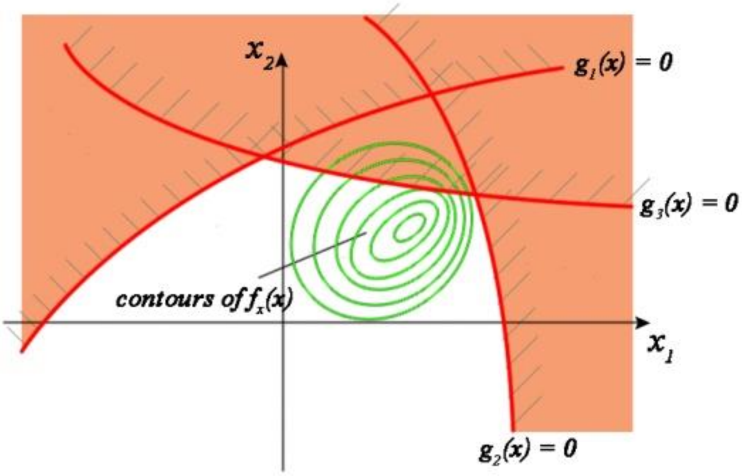
$$P_F = E[\bar{a}_z] = 1 - E[a_Z] = 1 - E[\Psi(a)]$$

Where, $\bar{a}_Z = 1 - a_Z$

In terms of component reliability analyses, event $\{a_{i=0}\}$, is equivalent to $\{g_i(x) \le 0\}$. In other words, the i^{th} component fails when the outcomes of the random variables in the problem define a point in the failure domain of the component. For a series system consisting of n components, the system fails if any of its components fail. Thus, the failure domain of a series system, Ω_z , is the union of the component failure domains and is given by:

$$\Omega_Z = \{\cup_{i=1}^{n} (g)X \quad 0 \le\}$$

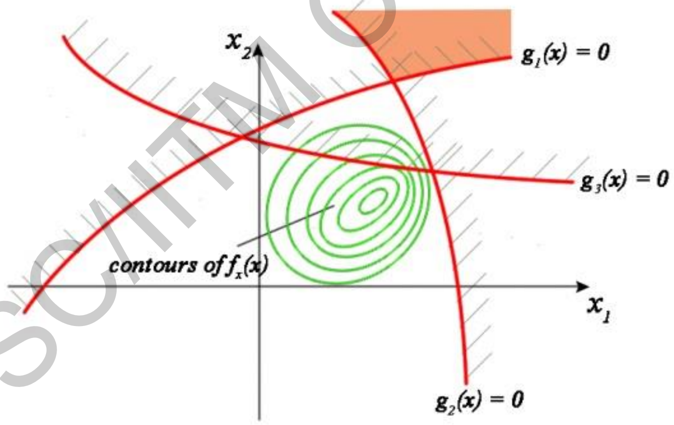
Figure below shows failure domains of a series system consisting of three components. In the Figure, hatching along the limit state surfaces indicates the region in which the corresponding limit state function is less than zero.



For a parallel system consisting of n components, the system fails if all of its components fail. The failure domain for a parallel system Ω_p is the intersection of the component failure domains and is given by:

$$\Omega_p = \cap_{i=1}^{n} \{g_i(x) \leq 0\}$$

Which is shaded in the Figure below:



For a general system, the failure domain, Ω_z is given by the cut set formulation:

$$\Omega_g = \cup_m \cap_{i \in c_m} \{g_i(x) \leq 0\}$$

Where C_m, is the m^{th} cut set. The safe domain for a general system, $\bar{\Omega}_g$ is defined by the path set formulation:

$$\bar{\Omega}_g = \cup_i \cap_{i \in p_i} \{g_i(x) > 0\},$$

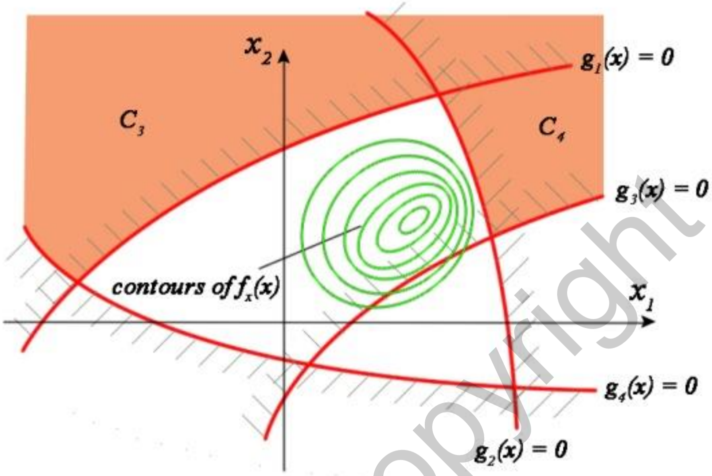
Where P_1 is the l^{th} path set. For the general system, failure domain is given by the union of the minimum cut sets C_4 = {2,3,4} and C_5= {1,4} identified for the system.

This failure domain is shaded in Figure below for some assumed component limit state surfaces. Now, problem of system reliability can be stated as:

$$P_F = \int_{\Omega_g} f(xdx)$$

Or alternatively, as given below:

$$1 - P_F = \int_{\overline{\Omega_R}} f(x)\, dx$$



4. Explain how first order estimate of system reliability can be made for series systems

In order to compute system reliability, one should compute probability of a union of events (series systems or subsystems) or probability of intersection of events (parallel systems or subsystems).

a) For a series system, following relationship holds good:

$$P_F = P(\cup_{i=1}^{n} \{g_i(x) \le 0\})$$

After applying to the appropriate transformation u = u(x) to the standard normal space, above equation can be modified as follows:

$$P_F \approx P(\cup_{i=1}^{n} \{G_i(u) \le 0\})$$

Where an approximation due to the mapping of non-normal variables to the standard normal space is introduced.

By Linearizing $G_i(u) = 0$ at the design point for the i^{th} limit state function by means of a Taylor series expansion, one can obtain the reliability index using the following relationship:

$G_i(u) \approx \nabla G^T(u_i{}^*)(u - u_i{}^*) = \left|\left|\nabla G_i(u_i{}^*)\right|\right|[-\alpha_i^T(u - u_i{}^*)] = \left|\left|\nabla G_i(u_i{}^*)\right|\right|[\beta_i - \alpha_i^T u]$

Where, u_i^* and β_i are the design point and reliability index, respectively for the i^{th} component (obtained from a FORM analysis of the component), α_i is the corresponding unit normal vector to the limit state surface $\mathrm{G_i}(u) = 0$ at u_i^*. Substituting these values, above equation takes a modified form as below:

$P_F \approx P[\cup_{i=1}^{n} \{(\beta_i - \alpha_{iu}^T) \leq 0\}]$

After dividing both sides of the inequality by the positive scalar $\left|\left|\nabla G_i(u_i^*)\right|\right|$ we now define

$Z_i = -\alpha_i^T \sim N(0,1)$

Which is a standard normal variable, as indicated Thus, the above equation can be re-written as:

$P_F \approx P(\cup_{i=1}^{n} \{Z_i \leq -\beta_i\})$

Using de Morgan's laws and making use of the rotational symmetry of the standard normal space, one can express probability of failure as below:

$P_F \approx P(\cup_{i=1}^{n} \{Z_i \leq -\beta_i\}) = 1 - P(\cap_{i=1}^{n} \{Z_i > -\beta_i\})$

$= 1 - P(\cap_{i=1}^{n} \{Z_i \leq -\beta_i\})$

$= 1 - \Phi_n(\beta, R_{ZZ})$

Where $\Phi_n(\beta, R_{ZZ})$is the joint normal CDF with Correlation matrix R_{ZZ} evaluated at $\beta = [\beta_1, \beta_2, \dots \dots . \beta_n]^T$

As variables $(\sigma_i = 1, i = 1,2 \dots \dots .. n)$ are standard normal variables, correlation matrix R_{ZZ} is identical to the covariance matrix.Σ_{zz} which can be obtained as below:

$$R_{ZZ} = \Sigma_{ZZ} = A\Sigma_{UU}A^T = AA^T$$

Where the i^{th}row of A is α_i^T and $\Sigma_{UU} = 1$ the identity matrix, due to the definition of standard normal variables u . Thus the off-diagonal terms in $R_{ZZ}$are

$\rho Z_i Z_j = \alpha_i^T \alpha_j$

The above equation quantifies correlation between failure modes i and j. Note that unlike component reliability analyses, in which the unit normal vector α is of secondary importance due to the rotational symmetry of the standard normal space, in system analyses, relative directions of the unit normal $\alpha_i$and $\alpha_j$plays a significant role.

5. Explain how first order estimate of system reliability can be made for parallel systems

For a parallel system, probability of failure is given by:

$$P_F = P(\cap_{i=1}^{n} \{g_i(x) \leq 0\})$$

Using a similar approach used in series system, Probability of failure can be approximated as below:

$$P_F = P(\cap_{i=1}^{n} \{g_i(x) \leq 0\}) \approx P(\cap_{i=1}^{n} \{G_i(u) \leq 0\})$$

$$\approx P(\cap_{i=1}^{n} \{(\beta_i - \alpha_i^T u) \leq 0\})$$

$$\approx P(\cap_{i=1}^{n} \{Z_i \leq -\beta_i\})$$

$$\approx \Phi_n(-\beta, R_{Zz})$$