Assignment 1

1. $a = 2, b = 1, c = 2, d = 1, e = 0, f = i$.
2. (a) Domain: R^2, Co-domain: R^3.

   (b) Range: <a_1,a_2>, span of a_1 and a_2. Null space = {0} in this case.

   (c) $A = \begin{bmatrix} -\frac{7}{2} & 3 \\ 9 & -6 \\ -\frac{5}{2} & 2 \end{bmatrix}$.

   (d) $y = Ax = \begin{bmatrix} -4 \\ 12 \\ -3 \end{bmatrix}$.

   (e) $\bar{A}\bar{x} = y, x = P\bar{x} \Longrightarrow \bar{A} = AP$.
3. (a) The null space is non-trivial, certainly containing $(x_1 - x_0)$ and its scalar multiples.

   (b) The set of pre-images is an infinite set, certainly containing (possibly as a subset) all vectors of the form $x_0 + \alpha(x_1 - x_0)$.
4. (a) $v_1 = \begin{Bmatrix} 0.82 \\ 0 \\ -0.41 \\ 0.41 \end{Bmatrix}$

   (b) $v_2 = \begin{Bmatrix} -0.21 \\ 0.64 \\ 0.27 \\ 0.69 \end{Bmatrix}$

   (c) $v_3 = \begin{Bmatrix} 0.2 \\ -0.59 \\ 0.72 \\ 0.33 \end{Bmatrix}$

   (d) $v_4 = \begin{Bmatrix} -0.50 \\ -0.50 \\ -0.50 \\ 0.50 \end{Bmatrix}$

   (e) Setting $v_1 = u_1/\|u_1\|$ and $l = 1$; for $k = 2,3, \dots, m$,
   $\tilde{v}_k = u_k - \sum_{j=1}^{l}(v_j^T u_k)v_j$; if $\tilde{v}_k \neq 0$, then $v_{l+1} = \tilde{v}_k/\|\tilde{v}_k\|$ and $l \leftarrow l + 1$.
5. (a) $C = \begin{bmatrix} \frac{\cos 10°}{5} & -15\sin 5° \\ -\frac{\sin 10°}{5} & 15\cos 5° \end{bmatrix}$.

   (b) $C^{-1} = \frac{1}{3\cos 15°}\begin{bmatrix} 15\cos 5° & 15\sin 5° \\ \frac{\sin 10°}{5} & \frac{\cos 10°}{5} \end{bmatrix}$.
6. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$.

   (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
7. Rank = 3,

   Nullity = 2,

   Range = $< q_1, q_2, q_3 >$,

Null space = $< n_1, n_2 >$, where $n_1 = \begin{Bmatrix} -0.7 \\ 0.1 \\ 2.1 \\ 1 \\ 0 \end{Bmatrix}, n_2 = \begin{Bmatrix} 0.1 \\ 0.7 \\ 2.7 \\ 0 \\ 1 \end{Bmatrix}$;

Particular solution: $\bar{x} = \begin{Bmatrix} 2.3 \\ -2.9 \\ -13.9 \\ 0 \\ 0 \end{Bmatrix}$,

General solution $x = \bar{x} + \alpha_1 n_1 + \alpha_2 n_2$.

8. $a = 5, b = 2, c = 1, d = 2, e = 2, f = -3, g = 1, h = -1, i = 3$.
9. $x_3 = 20/17, x_4 = 1$.
10. $Q = \begin{bmatrix} 0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 & -0.5 \end{bmatrix}$ and $R = \begin{bmatrix} 12 & 6 & 0 & 4 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Assignment 2:

1. $L^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{ad} & \frac{1}{d} & 0 \\ \frac{be-cd}{adf} & -\frac{e}{df} & \frac{1}{f} \end{bmatrix}$.
2. (a) Rank=1.
   (b) The formula is used to make small updates on an already computed inverse.
3. $\sum_{j=1}^{n} a_{ij} y_j , \sum_{j=1}^{m} a_{ji} x_j;$
   $Ay, A^T x;$
   $\sum_{j=1}^{n} (a_{ij} + a_{ji}) x_j ; (A + A^T)x.$
4. -
5. (a) $\begin{bmatrix} 1 + \frac{u_2}{2u_1} \\ 1 - \frac{u_2}{2u_1} \end{bmatrix}$.
   (b) $\sqrt{2 + u_1^2 + \sqrt{4 + u_1^4}}$ and $\frac{2+u_1^2+\sqrt{4+u_1^4}}{2u_1}$.
   (c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, 2.000025 and 200.005.
   (d) The solution is sensitive to small changes in u_1 and with u_2 it changes as $\delta x = 50 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \delta u_2$.
6. -
7. (a) Eigenvalues: -2, 3, 3; Eigenvectors: $\begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 3 \\ 4 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$.
   (b) Eigenvalues: 1, 2, 2; Eigenvectors: $\begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$.
8. $\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1} \lambda + a_n$.
   Significance: for every n-th degree polynomial, we can construct an $n \times n$ matrix, called its companion matrix, whose eigenvalues are the roots of the given polynomial.
9. (a) $\bar{A}_1 = \begin{bmatrix} \lambda_1 & b_1^T \\ 0 & A_1 \end{bmatrix}, b_1^T = q_1^T A \bar{Q}_1, A_1 = \bar{Q}_1^T A \bar{Q}_1$.
   (b) $\tilde{Q}_2 = [q_2 \; \bar{Q}_2], Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q}_2 \end{bmatrix}$.
   (c) For a real matrix A, having real eigenvalues, there exists an orthogonal matrix Q such that $Q^T AQ$ is upper triangular, with the eigenvalues at the diagonal.
10. Eigenvalue $\lambda = 28$ and corresponding eigenvector $v = \begin{Bmatrix} -0.41 \\ 0.41 \\ -0.82 \end{Bmatrix}$.

Assignment 3:

1. (a) Jordan form: $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (b) Diagonal form: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (c) Jordan form: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

   (d) Diagonal form: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$.

2. Deflated matrix: $\begin{bmatrix} 0.64 & 0.48 & 0 \\ 0.48 & 0.36 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

   other eigenvalues: 1,0;

   and corresponding eigenvectors: $\begin{Bmatrix} 0.8 \\ 0.6 \\ 0 \end{Bmatrix}$, $\begin{Bmatrix} -0.36 \\ 0.48 \\ 0.8 \end{Bmatrix}$ respectively.

3. Eigenvalues and eigenvectors:

   $0.62 + 0.785i, \begin{Bmatrix} -0.115 - 0.5i \\ -0.115 + 0.5i \\ 0.688 \end{Bmatrix}$;

   $0.62 - 0.785i, \begin{Bmatrix} -0.115 + 0.5i \\ -0.115 - 0.5i \\ 0.688 \end{Bmatrix}$;

   $1, \begin{Bmatrix} 0.688 \\ 0.688 \\ 0.299 \end{Bmatrix}$.

   $\det(Q) = 1$.

   Yes, Q represents a rotation.

   Plane orthogonal to the third eigenvector, or the plane of the first two eigenvectors, of which the real plane can be obtained with the basis $\left\{ \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

4. $\begin{bmatrix} 9.00 & 3.61 & 0 & 0 & 0 \\ 3.61 & 9.54 & 1.69 & 0 & 0 \\ 0 & 1.69 & 6.87 & -2.44 & 0 \\ 0 & 0 & -2.44 & 5.01 & -1.14 \\ 0 & 0 & 0 & -1.14 & 7.58 \end{bmatrix}$

5. $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. (Alternative solutions are possible).

6. (a) One eigenvalue is -1. All other eigenvalues are 1 with corresponding eigenvectors forming an eigenspace orthogonal to the eigenvector corresponding to the eigen value -1. Determinant = -1.

   (b) All eigenvalues have magnitude 1. There can be one or more unit $(+1)$ eigenvalue with real eigenvectors. Besides, on 2-d planes of rotation, there may be pairs of eigenvalues in the form of $e^{\pm i\theta}$ with associated eigenvectors in the form $u \pm iw$. In a special case of $\theta = \pi$, u and w separate out as individual (real) eigenvectors. Finally, there may remain a single negative eigenvalue -1 signifying reflection.

7. Five.

8. 5.81, 3.53, 1.66. Eigenvectors are the columns of cumulative product Q of the orthogonal matrices arising out of the QR iterations.
9. Eigenvalues: 8, 14.8, 5.8, -4.6, 10, 8 (appearing in this order).
10. Accurate eigenvalue=30.289, corresponding eigenvector = $\begin{Bmatrix} 0.38 \\ 0.53 \\ 0.55 \\ 0.52 \end{Bmatrix}$;

    The eigenvalue near to zero is 0.010 and the corresponding eigenvector is $\begin{Bmatrix} 0.83 \\ -0.50 \\ -0.21 \\ 0.12 \end{Bmatrix}$.

Assignment 4

1) (a) $\lambda_1 = 15625, \lambda_2 = 0.$ $v_1 = \begin{Bmatrix} 0.8 \\ -0.6 \end{Bmatrix}, v_2 = \begin{Bmatrix} 0.6 \\ 0.8 \end{Bmatrix}.$

   (b) $\Sigma = \begin{bmatrix} 125 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$

   (c) $U = \begin{bmatrix} 0.8 & 0 & 0.6 \\ 0.36 & 0.8 & -0.48 \\ -0.48 & 0.6 & 0.64 \end{bmatrix}$

   $(Note: Alternative\ solutions\ are\ possible).$

   (d) Basis for null space is v_2.

   (e) Basis for range space is $u_1 = \begin{Bmatrix} 0.8 \\ 0.36 \\ -0.48 \end{Bmatrix}.$

   (f) $V^T x = y\ and\ U^T b = c \Longrightarrow \Sigma y = c.$

2) Pseudoinverse solution:

   (a) $x_0 = V\Sigma^{\#}U^T b = \begin{Bmatrix} 0.16 \\ -0.12 \end{Bmatrix}.$

   (b) $E_0 = 25.$

   (c) $\Delta E = E - E_0 = (100\alpha - 75\beta)^2 \geq 0$

   (d) $\Delta\|x\|^2 = \|x\|^2 - \|x_0\|^2 = 25\left(\frac{\alpha}{3}\right)^2 = 25\left(\frac{\beta}{4}\right)^2 \geq 0.$

3) Ellipse: major radius 5.728 in direction $[0.9029\ 0.4298]^T$.

   Principal scaling effects: 5.728 on $[0.7882\ 0.6154]^T$ and $[-0.6154\ 0.7882]^T$.

   Eigen vectors: $[1\ 0]^T$ with eigenvalue 5 and $[-0.8944\ 0.4472]^T$ with eigenvalue 4.

4) 99.25% when only nine singular values are retained.

5) Not commutative.

6) Independent x,y,z: $\frac{\partial w}{\partial x} = 2x - 1.$

   Independent x,y,t: $\frac{\partial w}{\partial x} = 2x - 2.$

   Independent x,z,t: impossible.

7) Gradient = $\begin{bmatrix} 7.389 \\ 29.556 \end{bmatrix}$. (Four function values used).

   Hessian = $\begin{bmatrix} 8 & 44.25 \\ 44.25 & 148 \end{bmatrix}$. (Nine function values used).

8) $5x_1^2 + 3x_1x_2 - 20x_1 - 4x_2 + 28$; hyperbolic contours.

9) $\frac{5}{6}\sqrt{6}$ feet or 2.04 feet.

10) -

Assignment 5:

1. Potential function: $\frac{c}{\|r-r_0\|}$.
2. $div\ E = \frac{1}{\epsilon}Q,\ div\ B = 0, curl\ E = -\frac{\partial B}{\partial t}, curl\ B = \mu\epsilon\frac{\partial E}{\partial t}$.
   $\frac{\partial^2 E}{\partial t^2} = c^2\nabla^2 E$ and $\frac{\partial^2 B}{\partial t^2} = c^2\nabla^2 B$.
3. oh
4. Eigenvalues: 2.6185, -2.3032, 1.3028, 0.3820. Values of the polynomial at these values of x: 0.0067, 0.0202, -0.0001, 0.0002 (respectively).
5. (a) $\frac{\partial \tau}{\partial p} = up - v, \frac{\partial \tau}{\partial q} = -u, \frac{\partial s}{\partial p} = uq$, and $\frac{\partial s}{\partial q} - v$.
   (b) $J = \begin{bmatrix} up - v & -u \\ uq & -v \end{bmatrix}, \begin{bmatrix} p_{k+1} \\ q_{k+1} \end{bmatrix} = \begin{bmatrix} p_k \\ q_k \end{bmatrix} - \frac{1}{\Delta_k}\begin{bmatrix} -v_k & u_k \\ -u_k q_k & u_k p_k - v_k \end{bmatrix}\begin{bmatrix} r_k \\ s_k \end{bmatrix}$, for $\Delta_k = v_k^2 - u_k p_k v_k + u_k^2 q_k \neq 0$.
   (c) Roots: $4.37, -1.37, 8.77, 0.23, 3.5 \pm 2.4i$.
6. 0.57. The solution is unique.
7. 0.17.
8. $[0.88\ 0.68\ 1.33]^T$.
9. $\phi\left(\frac{1}{2}\right) = 0\ maximum, \phi\left(\frac{1}{2} \pm \frac{1}{5}\right) = -\frac{1}{25}\ minima.$
   Other features: symmetry about $x = \frac{1}{2}$, zero-crossing at $x = \frac{1}{2} \pm \frac{\sqrt{2}}{5}$, etc.
10. Minima: $(0,0)$, $(\pm\sqrt{3}, \mp\sqrt{3})$; Saddle points: $(\pm 1, \mp 1)$; no maximum point.

Assignment 6:

1. [1.90, 1.92].
2. $x^* = 2.5$.
3. (a) $1, \begin{Bmatrix} -2 \\ 0 \\ 0 \end{Bmatrix}, 24\alpha^2 - 4\alpha + 1$.

   (b) $0.0833, (0.166,0,0)$.

   (c) and (d):

k	x_k	$f(x_k)$	$g(x_k)$	α_k	x_{k+1}	d_{k+1}
0	$(0,0,0)$	1	$(-2,0,0)$	0.083	$(0.166,0,0)$	1.64
1	$(0.166,0,0)$	0.833	$(0,-1,-0.667)$	0.191	$(0.166,0.191,0.127)$	1.45
2	$(0.166,0.191,0.127)$	0.695	$(-1.657,0.020,-0.029)$	0.083	$(0.305,0.190,0.130)$	1.38

4. (1.2, 1.2, 3.4); $f^* = -12.4$.
5. $x_0 = [2\ 2]^T, f(x_0) = 5; x_1 = [1.8\ 3.2]^T, f(x_1) = 0.6416$.
6. (a) $x = 3.9094, y = 0.6969$.

   (b) $x = -0.5652, y = 0.570, z = 1.5652$.
7. For the single-variable problem in λ with an embedded 4-variable linear least square problem, starting with bounds $-5 \le \lambda \le 5$, the result is $[91.05, 5.30, -1.80, -71.08, -0.51]^T$, with error value 0.1961.
8. KKT conditions:

$$2\boldsymbol{C}\boldsymbol{C}^{\boldsymbol{T}}x + 2\boldsymbol{C}^{\boldsymbol{T}}(\boldsymbol{a} - \boldsymbol{p}) + \mu_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \boldsymbol{\nu} = 0$$

   where $\boldsymbol{C} = \boldsymbol{b} - \boldsymbol{ap} - \boldsymbol{qp} - \boldsymbol{r}$
9. $\phi(\mu) = -\frac{5}{4}\mu^2 + 7\mu;\ \mu_{max} = 9.8; x_{min} = [0.2\ 1.6]^T, f_{min} = 9.8$.
10. $\phi(\lambda) = \lambda_2 L - l \sum_{i=1}^{n} \sqrt{(c_i + \lambda_1)^2 + \lambda_2^2}$.

    Solution: $\lambda_1 = -10, \lambda_2 = 6.76, \phi_{max} = -665.5$.

Assignment 7:

1. -
2. $W = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.
3. Hermite polynomial: $x(t) = 0.0785t^2 - 0.0045t^3 + 0.000116t^4 - 1.13 \times 10^{-6}t^5$.
4. (a) $p(s) = \frac{1}{2}s(s-1)f_0 - (s-1)(s+1)f_1 + \frac{1}{2}s(s+1)f_2$.

   (b) $\frac{h}{3}[f_0 + 4f_1 + f_2]$.

   (c) $-\frac{h^5}{90}f^{iv}(x_1) + higher\ order\ terms$
5. 2.495151.
6. $I_4 = 0.697023810$
   $I_8 = 0.694121850$
   $I_{16} = 0.693391202$
   $I_R = 0.693147194$.
7. 213 cubic units.
8. -
9. -
10. -

Assignment 8:

1. -
2. $f''(0) = 0.4696$.
3. $y(x) = (1-w)y^{(0)}(x) + wy^{(1)}(x)$ is the solution, where $y^{(0)}(x)$ and $y^{(1)}(x)$ are two solutions of the ODE with $y(a) = p_1$ and arbitrary values of $y'(a)$, and w is determined from $(1-w)y^{(0)}(b) + wy^{(1)}(b) = p_2$.
4. $y(1) = 0.20, y(1.6) = 1.18$ and $y(2) = 2.81$.
5. (a) $x^2 \cos y + x^4 = c$

   (b) $x^2 - 3xy - y^2 + 2x - 5y + c = 0$

   (c) $3\ln(14x + 21y + 22) = -7x + 14y + k$

   (d) $y^2 = \frac{x}{2}e^x + cxe^{-x}$

   (e) $x^y = e^{-\frac{x^3}{3}}$

   (f) $e^y \cos(x + c) = k$

   (g) $y = \frac{A}{1-e^{Ax+B}}$
6. (a) Solution y=cx is not unique; Lipschitz condition is not satisfied at (0,0).

   (b) Lipschitz condition is satisfied, IVP is well-posed; but $\frac{\partial f}{\partial y}$ does not exist for $y = 0$.
7. $x^2 + 2y^2 = c^2$. [Family of ellipses.]

   $x^2 + ny^2 = c$. [Family of central conics.]
8. (a) $\frac{dA}{dx} = D + rA$

   (b) $\frac{D}{r}(e^{rt} - 1)$

   (c) Approximately Rs. 59,861.
9. (a) $y = e^{-x}(c_1 \cos x + c_2 \sin x)$

   (b) $y = e^{-\frac{x}{2}}$

   (c) $y = 3x^2 \ln x$
10. $y(x) = 1 + x - \frac{1}{(2)(3)}x^3 - \frac{1}{(3)(4)}x^4 + \frac{1}{(2)(3)(5)(6)}x^6 + \frac{1}{(3)(4)(6)(7)}x^7 - \frac{1}{(2)(3)(5)(6)(8)(9)}x^9 + \cdots$

Assignment 9:

1. (a): $y = x^4 + x$.
   (b) $y = c_1 \cos 2x + c_2 \sin 2x + 4x + \frac{8}{5}e^x$.
   (c) $y = c_1 x + c_2 x^2 - x \cos x$.
2. (a) $y_1(x) = x$.
   (b) $x(x+1)(2x+1)u'' + 2(3x^2 + 3x + 1)u' = 0$.
   (c) $u = \frac{1}{x} - \frac{1}{x+1}$.
   (d) $y(x) = c_1 x + \frac{c_2}{x+1}$. Not valid for interval [-5,5].
3. $y(x) = -\frac{37x}{24} + \frac{23}{12(x+1)} + \frac{x^2(4x+3)}{6(x+1)}$.
4. $y(t) = \frac{Ate^{-ct}}{2\omega} \sin \omega t$.
5. Case I: $\nu = 0$.
   (a) $y = x - \sin x$
   (b) $y = \frac{4}{\pi}\left(1 + \frac{1}{\sqrt{2}}\right) x - \sin x$
   (c) $y = -\sin x$
   (d) $y = \frac{x}{2\pi} - \sin x$
   (unique for all the sets of conditions).
   Case II: $\nu = 1$ or $\nu = -1$.
   (a) $y = \frac{1}{2}(\sin x - x \cos x)$
   (unique)
   (b) $y = \left(\sqrt{2} + \frac{\pi}{8}\right) \sin x - \frac{1}{2} x \cos x$
   (unique)
   (c) and (d) no solution exists.
   Case III: $\nu \neq 0, \nu \neq 1, \nu \neq -1$.
   (a) $y = \frac{1}{\nu^2 - 1}\left(\sin x - \frac{\sin \nu x}{\nu}\right)$
   (unique)
   (b) no solution exists for $\nu = \pm 4, \pm 8, \pm 12, \ldots$
   otherwise, $y = \csc\frac{\nu\pi}{4}\left[1 - \frac{1}{\sqrt{2}(\nu^2-1)}\right] \sin \nu x + \frac{\sin x}{\nu^2 - 1}$ (unique)
   (c) infinite solutions $y = B \sin \nu x + \frac{\sin x}{\nu^2 - 1}$ with arbitrary B for $\nu = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$
   otherwise $y = \frac{\sin x}{\nu^2 - 1}$ (unique)
   (d) no solution exists for $\nu = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$
   otherwise $y = \csc 2\nu\pi \sin \nu x + \frac{\sin x}{\nu^2 - 1}$ (unique).
6. (a) $y(x) = \left(c_1 + c_2 x + c_3 x^2 + \frac{8}{105} x^{\frac{7}{2}}\right) e^{2x}$.
   (b) $y(x) = \sin x + \sin 3x + 2 \sinh x$.
7. $x(t) = \begin{bmatrix} 1 & t-1 & \frac{t^2}{2} - t \\ -1 & -t & -\frac{t^2}{2} \\ 2 & 2t-1 & t^2 - t + 1 \end{bmatrix} \begin{bmatrix} c_1 e^t + \frac{1}{2}t^2 + t + 1 - \left(\frac{1}{4}t^2 + \frac{3}{4}t - \frac{1}{8}\right) e^{-t} \\ c_2 e^t - t - 1 + \left(\frac{1}{2}t + \frac{3}{4}\right) e^{-t} \\ c_3 e^t + 1 - \frac{1}{2} e^{-t} \end{bmatrix}$

8. (a) $x' = \begin{bmatrix} \phi' \\ \phi'' \\ I'_a \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{(k_t x_3 = \mu x_2)}{J} \\ \frac{V - R_a x_3 - k_b x_2}{L_a} \end{bmatrix}$

   (b) In the revised model, state vector $y = [\omega\ I_a]^T$, with $\omega = \phi'$, and

   $$y' = \begin{bmatrix} \omega' \\ I'_a \end{bmatrix} = \begin{bmatrix} -\frac{\mu}{J} & \frac{k_t}{J} \\ -\frac{k_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} y + \begin{bmatrix} 0 \\ \frac{V}{L_a} \end{bmatrix}$$

   (c) Steady state: $y_0 = [k_t\ \mu]^T V/\Delta$, where $\Delta = k_b k_t + \mu R_a$.

   (d) $z = y - y_0$, $z' = Bz$, where $B = \begin{bmatrix} -\mu/J & k_t/J \\ -k_b/L_a & -R_a/L_a \end{bmatrix}$.

9. Critical points: origin with eigenvalues $-0.5 \pm 0.866i$ is a stable spiral, and points $(\pm 1{,}0)$ with eigenvalues -2,1 are (unstable) saddle points.

10. -

Assignment 10:

1. (a) $y(x) = a_0\left[1 + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{3}{6!}x^6 + \cdots\right] + a_1 x + \left[\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{7}{40}x^5 + \frac{19}{240}x^6 + \cdots\right]$

   (b) $y(x) = c_1 y_1(x) + c_2 y_2(x), y_1(x) = \frac{1}{x^2}\sum_{k=0}^{\infty}\frac{(-x)^k}{(k!)^2}, y_2(x) = y_1(x)\ln x + \frac{2}{x} - \frac{3}{4} + \frac{11}{108}x + \cdots$
2. -
3. $y_1(x) = 1 + \frac{(-k^2)}{2!}x^2 + \frac{(2^2-k^2)(-k^2)}{4!}x^4 + \frac{(4^2-k^2)(2^2-k^2)(-k^2)}{6!}x^6 + \cdots$

   $y_2(x) = x + \frac{(1-k^2)}{3!}x^3 + \frac{(3^2-k^2)(1-k^2)}{5!}x^5 + \frac{(5^2-k^2)(3^2-k^2)(1-k^2)}{7!}x^7 + \cdots$

   For integer k, one of the solutions is a polynomial in the form

   $a_k\left[x^k - \frac{k}{1!2^2}x^{k-2} + \frac{k(k-3)}{2!2^4}x^{k-4} - \frac{k(k-4)(k-5)}{3!2^6}x^{k-6} + \cdots\right]$.
4. $p(x) = e^{-x}$.
5. (a) $y = \cos(n\cos^{-1}x), y = \sin(n\cos^{-1}x)$.

   (b) $T_n(x) = \cos(n\cos^{-1}x)$: $T_0(x) = 1,\ T_1(x) = x, T_{k+1} = 2xT_k(x) - T_{k-1}(x)$.

   (c) Weight function: $\frac{1}{\sqrt{1-x^2}}$.

   (d) -

   (e) $\|T_0(x)\| = \sqrt{\pi}$ and $\|T_n(x)\| = \sqrt{\frac{\pi}{2}}$ for $n = 1, 2, 3 \ldots$

   (f) -
6. $\psi_3(x) = 6x^2 - 6x + 1$.
7. Cosine series:

   $f_c = 0.857 - 0.798\cos\frac{\pi x}{2} - 0.345\cos\pi x + 0.351\cos\frac{3\pi x}{2} + 0.053\cos 2\pi x + \cdots$

   $f_c(0) \approx 0.0566, f_c(1) \approx 1.3266$ and $f_2(2) \approx 1.0089$.

   Sine series:

   $f_s(x) = 1.235\sin\frac{\pi x}{2} - 0.843\sin\pi x + 0.078\sin\frac{3\pi x}{2} + 0.056\sin 2\pi x + \cdots$

   $f_s(0) \approx 0, f_s(1) \approx 1.2785$ and $f_s(2) \approx 0$.
8. $I(t) = \sum_{k=1, k\ odd}^{\infty}\frac{80}{k^2\pi[(10-k^2)^2+(10k)^2]}[(10-k^2)\cos kt + 10k\sin kt]$.
9. Taylor's series approximation:

   $e_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$,

   maximum deviation 0.0516 at $x = 1$.

   Legendre series approximation:

   $e_P(x) = 1.2660 + 1.1302T_1(x) + 0.2714T_2(x) + 0.0442T_3(x)$,

   maximum deviation 0.0112 at $x = 1$.

   Chebyshev series approximation:

   $e_T(x) = 1.2660 + 1.1302T_1(x) + 0.2714T_2(x) + 0.0442T_3(x)$,

   maximum deviation 0.0064 at $x = 1$.
10. Chebyshev accuracy points: $x_j = 5\cos\left(\frac{2j+1}{18}\pi\right)$ for $j = 0,1,2,\ldots,8$.

Assignment 11:

1. $u(x,y) = \sum_i U_i x^{k_i} e^{-y^2/k_i}$.
2. $u(x,y) = f_1(y-x) + f_2(y-2x) + (y-x)(y-2x)^2 - \frac{3}{2}(y-x)^2(y-2x)$,
   or, $u(x,y) = f_1(y-x) + f_2(y-2x) - \frac{1}{2}(y^3 - 2xy^2 - x^2y + 2x^3)$.
3. $y(x,t) = \frac{5}{3\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(\pi e)}{n^2-e^2}\sin(nx)\sin(1.2nt)$.
4. $u(x,y,t) = k\sin\pi x\sin\pi y\cos(\sqrt{2}\pi t)$.
5. $\frac{d}{dz}(\cos z) = -\sin z$ and $\frac{d}{dz}(\sin z) = \cos z$.
6. –
7. (a) $w(z) = \frac{iz}{z-1}$
   (b) $\theta = k\frac{\pi}{4}$ maps to $\tan k\frac{\pi}{4} = \frac{u}{u^2+v(v-1)}$, which are circular arcs.
8. $2\pi i\rho^2$.
9. $\int z^n\,dz = \frac{z^{n+1}}{n+1}$ except for $n = -1$.
   For $n = -1$, $\int \frac{1}{z}dz = 2\pi i$.
10. $(a)\ \pi e^{2i}$.
    $(b) \pi(8i - 5)$.

Assignment 12:

1. $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$, radius of convergence = 1.
2. There are two possible Laurent's series, valid in different regions of convergence:
   (a) $\frac{z^2+1}{z^3-4z} = -\frac{1}{4z} - \frac{5z}{4^2} - \frac{5z^3}{4^3} - \frac{5z^5}{4^4} - \frac{5z^7}{4^5} - \cdots$ for $0 < |z| < 2$, and
   (b) $\frac{z^2+1}{z^3-4z} = \frac{1}{z} + \frac{5}{z^3} + \frac{5\cdot 4}{z^5} + \frac{5\cdot 4^2}{z^7} + \cdots$ for $2 < |z| < \infty$.
3. Zeroes of $\sin z$: $n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$, all real and isolated.
   Zeroes of $\sin\frac{1}{z}$: $\frac{1}{n\pi}$ for $n = \pm 1, \pm 2, \ldots$, all real and isolated, apart from $z = 0$.
4. At the origin, $e^{1/z}$ has an isolated essential singularity, and no zero in its neighbourhood. Arbitrarily close to this point, the function can assume any given (non-zero) value. For arbitrary complex number $Ae^{i\phi}$, $z = \frac{1}{\sqrt{(\ln A)^2 + (\phi)^2}} e^{i\tan^{-1}(-\phi/\ln A)}$ satisfies $e^{1/z} = Ae^{i\phi}$. To make $|z|$arbitrarily small, one can add $2k\pi$ to ϕ.
5. $2\pi i$.
6. (a) $s(\alpha, \beta) = \frac{m}{6}(\alpha^2 + \beta^2)T^3 + \frac{m(a^2+b^2)}{2T} + mg\left(\frac{\beta T^3}{6} - \frac{bT}{2}\right)$.
   (b) $\alpha = 0, \beta = -\frac{g}{2}$. Consequently, $x(t) = at/T$, $y(t) = -\frac{g}{2}t(t-T) + \frac{bt}{T}$; $\dot{x}(t) = a/T$, $\dot{y}(t) = -\frac{g}{2}(2t - T) + \frac{b}{T}$; $\ddot{x}(t) = 0, \ddot{y}(t) = -g$.
   (c) -
7. $f(x) \approx 0.2348x^2 - 0.0041x^3 + 0.0089x^4 - 0.0089x^5 + 0.0024x^6 - 0.0003x^7 + 0.00001x^8$.