Mathematical Methods in Engineering and Science

IIT Kanpur

Assignment Policy of the Course

- Assignments of this course are of the traditional problem-solving type. Not only the result, but also the approach will be considered for evaluation.
- Submissions are invariably to be in the PDF format. PDF version of scanned copy of neat handwritten work is acceptable. Brevity and to the point solutions will be rewarded. Illegible writing, sloppy work and beating around the bush will be penalized.
- For many segments of the course, a programming background will be useful, though it is not essential.
- For some problems in the assignments, programming will be needed to get the complete solution. But, the programme does not need to be submitted.
  Those students who do not have a programming background may conduct the intial few (2 to 5) steps or iterations manually and indicate through what repititive and automated procedure a programme would need to proceed in order to get the final complete solution. Such solutions will be considered for partial credit.
- There may also be some problems, for which programming may not be needed as such, but may be useful in solving the problem systematically. Students having a programming background are encouraged to take advantage of this.

MMES (2017) Assignment 9

(Full marks = 100)

1. Solve the following differential equations.

   (a) $xy'' + y' = 16x^3 + 1;\ y(1) = 2,\ y'(1) = 5.$

   (b) $y'' + 4y = 16x + 8e^x.$

   (c) $(x^2D^2 - 2xD + 2)y = x^3 \cos x.$

2. Consider the differential equation

$$(2x^2 + 3x + 1)y'' + 2xy' - 2y = 0.$$

   (a) Identify one non-trivial solution $y_1(x)$ by inspection.

   (b) Considering a second linearly independent solution as $u(x)y_1(x)$, set up a differential equation for the function $u(x)$.

   (c) Solve this differential equation to determine $u(x)$.

   (d) Hence, develop the general solution of the given differential equation. Is this valid for an interval $[-5, 5]$ of x?

3. Solve the boundary value problem

$$(2x^2 + 3x + 1)y'' + 2xy' - 2y = 4x^2 + 4x + 1;\quad y(1) = 0,\ y(2) = 0\ .$$

   [Use the results of the previous problem.]

4. Show that a solution of the IVP $\ddot{y} + 2c\dot{y} + (c^2 + \omega^2)y = Ae^{-ct}\cos\omega t,\quad y(0) = 0,\ \dot{y}(0) = 0$ represents an oscillation with a variable amplitude $\frac{A}{2\omega}te^{-ct}$. Analyze the response for amplitude, frequency, phase and maximum amplitude for small magnitude of c and also for $c \to 0$.

5. Find the solution(s) of $y'' + \nu^2 y = \sin x$ with the following sets of conditions, whenever possible:
   (a) $y(0) = 0,\ y'(0) = 0,$ (b) $y(0) = 0,\ y(\pi/4) = 1,$
   (c) $y(0) = 0,\ y(2\pi) = 0,$ and (d) $y(0) = 0,\ y(2\pi) = 1.$

6. Solve the following differential equations.

   (a) $y''' - 6y'' + 12y' - 8y = \sqrt{x}e^{2x}.$

   (b) $y^{iv} + 10y'' + 9y = 40\sinh x,\quad y(0) = 0,\ y'(0) = 6,\ y''(0) = 0,\ y'''(0) = -26.$

7. Solve the following system of differential equations:

$$x_1' = x_1 - 2x_2 - x_3,\ x_2' = x_1 + 2x_2 + e^{-t}, x_3' = -x_1 - 3x_2 - 1\ .$$

8. In a DC motor, the shaft angle (ϕ), the torque (τ), the armature current (I_a) and the voltage (V) are related through the dynamic equation, circuit equation and electro-mechanical coupling equation as

$$J\frac{d^2\phi}{dt^2} + \mu\frac{d\phi}{dt} = \tau \ , \quad L_a\frac{dI_a}{dt} + R_a I_a + k_b\frac{d\phi}{dt} = V \ , \quad \tau = k_t I_a \ ;$$

where motor inertia (J), viscous friction coefficient (μ), armature inductance (L_a), armature resistance (R_a), torque constant (k_t) and back emf constant (k_b) are constant system parameters.

(a) Develop the state-space equation for the motor with $\mathbf{x} = [\phi \ \ \frac{d\phi}{dt} \ \ I_a]^T$ as the state vector.

(b) Note the disadvantage of this system modelling from mathematical and engineering (physical) standpoints. Hence, revise the modelling with a *reduction* of the state space.

(c) From this reduced model, find the operating condition or the so-called steady state behaviour of the motor.

(d) Now, shift the dynamic model to this steady state to analyze the fluctuations of operation around this state.

(e) What are the different features of the phase portrait that may emerge at this operating point?

9. Develop and discuss the phase portrait of the damped nonlinear spring modelled by the equation $\ddot{x} + \dot{x} + x - x^3 = 0$.

10. Consider two species whose survival depends on their mutual cooperation. An example would be a species of bee that feeds primarily on the nectar of one plant species and simultaneously pollinates that plant, which has no other substantial means of pollination. One simple model of this *mutualism* is given by the autonomous system

$$\frac{dx}{dt} = -ax + bxy, \qquad \frac{dy}{dt} = -my + nxy.$$

(a) What assumptions are implicitly being made about the evolution of each species in the absence of cooperation? Interpret the constants a, b, m and n in terms of the problem.

(b) What are the equilibrium points?

(c) Analyze the system and sketch the phase portrait.

(d) Interpret the outcomes predicted by the phase portrait.