Mathematical Methods in Engineering and Science

IIT Kanpur

Assignment Policy of the Course

- Assignments of this course are of the traditional problem-solving type. Not only the result, but also the approach will be considered for evaluation.
- Submissions are invariably to be in the PDF format. PDF version of scanned copy of neat handwritten work is acceptable. Brevity and to the point solutions will be rewarded. Illegible writing, sloppy work and beating around the bush will be penalized.
- For many segments of the course, a programming background will be useful, though it is not essential.
- For some problems in the assignments, programming will be needed to get the complete solution. But, the programme does not need to be submitted.
  Those students who do not have a programming background may conduct the intial few (2 to 5) steps or iterations manually and indicate through what repititive and automated procedure a programme would need to proceed in order to get the final complete solution. Such solutions will be considered for partial credit.
- There may also be some problems, for which programming may not be needed as such, but may be useful in solving the problem systematically. Students having a programming background are encouraged to take advantage of this.

MMES (2017) Assignment 4

(Full marks = 100)

1. Consider the matrix
$$\mathbf{A} = \begin{bmatrix} 80 & -60 \\ 36 & -27 \\ -48 & 36 \end{bmatrix}.$$

   (a) Construct $\mathbf{A}^T\mathbf{A}$ and determine its eigenvalues λ_1, λ_2 (number them in descending order, for convenience) and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, as an orthonormal basis of R^2.

   (b) Define $\sigma_k = \sqrt{\lambda_k}$, form a diagonal matrix with σ_1 and σ_2 as the diagonal elements and extend it (with additional zeros) to a matrix Σ of the same size as $\mathbf{A}$.

   (c) Assemble the eigenvectors into an orthogonal matrix as $\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2]$ and find any orthogonal matrix $\mathbf{U}$ satisfying $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$.

   (d) Identify the null space of $\mathbf{A}$ in terms of columns of $\mathbf{V}$.

   (e) Identify the range space of $\mathbf{A}$ in terms of columns of $\mathbf{U}$.

   (f) How does a system of equations $\mathbf{Ax} = \mathbf{b}$ transform if the bases for the domain and the co-domain of $\mathbf{A}$ change to $\mathbf{V}$ and $\mathbf{U}$, respectively?

2. One possible singular value decomposition of the 3×2 matrix $\mathbf{A}$ in the above exercise is given by
$$\mathbf{A} = \mathbf{U\Sigma V}^T = \begin{bmatrix} 0.80 & 0 & 0.6 \\ 0.36 & 0.8 & -0.48 \\ -0.48 & 0.6 & 0.64 \end{bmatrix} \begin{bmatrix} 125 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}^T.$$

   (a) Find the pseudoinverse solution for the system $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{b} = [20 \ 13 \ -9]^T$ and denote this solution by $\mathbf{x}_0$.

   (b) Find $\|\mathbf{x}_0\|^2$ and $E_0 = \|\mathbf{Ax}_0 - \mathbf{b}\|^2$.

   (c) For $\mathbf{x} = \mathbf{x}_0 + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, show that $\Delta E = \|\mathbf{Ax} - \mathbf{b}\|^2 - \|\mathbf{Ax}_0 - \mathbf{b}\|^2 \geq 0$.

   (d) For $\Delta E = 0$, show that $\Delta\|\mathbf{x}\|^2 = \|\mathbf{x}\|^2 - \|\mathbf{x}_0\|^2 \geq 0$.

3. On the x_1–x_2 plane, sketch the ellipse that represents the image of the unit circle $x_1^2 + x_2^2 = 1$, through the linear transformation with the matrix
$$\mathbf{T} = \begin{bmatrix} 5 & 2 \\ 0 & 4 \end{bmatrix},$$
and show the major and minor axes. Find out the largest and least scaling effects of the matrix from the sketch and corroborate the results through computation of the SVD. What are the eigenvectors of $\mathbf{T}$?

4. In a large-scale problem, you encounter a particular 2000×3000 real matrix which requires the storage space of 6×10^6 real numbers. What approximate percentage saving of storage can you achieve without compromising accuracy badly, utilizing the fact that the singular values of the matrix, in descending order, are 74, 52, 37, 22, 22, 8, 8, 8, 4, 0.03, 0.02, 0.02, 0.005, 0.002, 0.0006 etc?

5. Let vector $\mathbf{p}$ denote the position vector of a general point (particle) belonging to a rigid body, in some fixed frame of reference. A rigid body motion is defined by a rotation (matrix) $\Re$ and a translation (vector) $\mathbf{r}$ such that the transformed location of $\mathbf{p}$ due to the motion is given by $\mathbf{q} = \Re\mathbf{p} + \mathbf{r}$. Show that all such rigid body motions form a group. Is this group commutative? [*Hint:* Consider the set of structures in the form of the pair $(\Re, \mathbf{r})$.]

6. From the relationships $x^2 - y^2 + 4z + t - w = 0$ and $x + 2z + t + 2 = 0$, we want to extract w and one of the variables from y, z and t as functions of x and the two other variables. In each of these cases, find $\frac{\partial w}{\partial x}$.

7. A function $f(\mathbf{x})$ of two variables has been evaluated at the following points.

$$\begin{array}{lll} f(1.999, 0.999) = 7.352232, & f(1.999, 1) = 7.381671, & f(1.999, 1.001) = 7.411257; \\ f(2, 0.999) = 7.359574, & f(2, 1) = 7.389056, & f(2, 1.001) = 7.418686; \\ f(2.001, 0.999) = 7.366922, & f(2.001, 1) = 7.396449, & f(2.001, 1.001) = 7.426124. \end{array}$$

Find out the gradient and Hessian of the function at the point (2,1). How many function values did you have to *use* for each of them?

8. Plot contours of the function $f(\mathbf{x}) = x_1^2 x_2 - x_1 x_2 + 8$, in the region $0 < x_1 < 3, \;\; 0 < x_2 < 10$. Develop a quadratic approximation of the function around (2,5) and superimpose its contours with those of $f(\mathbf{x})$. Are the contour curves of this quadratic approximation elliptic, parabolic or hyperbolic?

9. In a 10 feet $\times$ 10 feet $\times$ 10 feet children's playroom, a strong rope has been fixed by nails at the centres of two adjacent walls. From the top corner between these walls to the farthest corner of the room, a narrow wooden plank, with grooves to climb, is placed. A child, climbing along the plank, wants to jump and get hold of the rope. Determine the minimum distance the child has to scale between the plank and the rope.

10. "The vector functions

$$\begin{aligned} \mathbf{r}_1(t) &= 3\cos t\;\mathbf{i} + 3\sin t\;\mathbf{j}, \quad t \geq 0 \qquad \text{and} \\ \mathbf{r}_2(t) &= (2\sin 2t + 2\cos 2t + 1)\mathbf{i} + (2\sin 2t - \cos 2t + 2)\mathbf{j} + (\sin 2t - 2\cos 2t + 4)\mathbf{k}, \quad t \geq \pi/4 \end{aligned}$$

essentially represent the same curve". Confirm or refute this statement with valid arguments.