Mathematical Methods in Engineering and Science

IIT Kanpur

Assignment Policy of the Course

- Assignments of this course are of the traditional problem-solving type. Not only the result, but also the approach will be considered for evaluation.
- Submissions are invariably to be in the PDF format. PDF version of scanned copy of neat handwritten work is acceptable. Brevity and to the point solutions will be rewarded. Illegible writing, sloppy work and beating around the bush will be penalized.
- For many segments of the course, a programming background will be useful, though it is not essential.
- For some problems in the assignments, programming will be needed to get the complete solution. But, the programme does not need to be submitted.
  Those students who do not have a programming background may conduct the intial few (2 to 5) steps or iterations manually and indicate through what repititive and automated procedure a programme would need to proceed in order to get the final complete solution. Such solutions will be considered for partial credit.
- There may also be some problems, for which programming may not be needed as such, but may be useful in solving the problem systematically. Students having a programming background are encouraged to take advantage of this.

MMES (2017) Assignment 2

(Full marks = 100)

1. Show that the inverse of a triangular matrix is also triangular. Invert the matrix

$$\mathbf{L} = \begin{bmatrix} a & & \\ b & d & \\ c & e & f \end{bmatrix}$$

and develop a generalized algorithm for inverting an $n \times n$ non-singular triangular matrix.

2. For an invertible matrix $\mathbf{A} \in R^{n\times n}$ and non-zero column vectors $\mathbf{u}, \mathbf{v} \in R^n$,

   (a) find out the rank of the matrix $\mathbf{u}\mathbf{v}^T$, and

   (b) prove the following identity **(Sherman-Morrison formula)** and comment on its utility.

$$(\mathbf{A} - \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{u}(1 - \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})^{-1}\mathbf{v}^T\mathbf{A}^{-1}.$$

3. For a bilinear form $p(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{A}\mathbf{y}$ of two vector variables $\mathbf{x} \in R^m$ and $\mathbf{y} \in R^n$, find out $\frac{\partial p}{\partial x_i}$, $\frac{\partial p}{\partial y_i}$; and hence the vector gradients $\frac{\partial p}{\partial \mathbf{x}}$ and $\frac{\partial p}{\partial \mathbf{y}}$. As a corollary, derive the partial derivative $\frac{\partial q}{\partial x_i}$ and vector gradient of a quadratic form $q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$.

4. For a real invertible matrix $\mathbf{A}$, show that $\mathbf{A}^T\mathbf{A}$ is positive definite. Further, show that, for $\nu > 0$, $\mathbf{A}^T\mathbf{A} + \nu^2\mathbf{I}$ is positive definite for any real matrix $\mathbf{A}$.

5. Let

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 + u_1 & 1 - u_1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 2 \\ 2 + u_2 \end{bmatrix}.$$

   (a) Find the solution $\mathbf{x}$ of the system $\mathbf{Q}\mathbf{x} = \mathbf{d}$ as a function of u_1 and u_2.

   (b) Find the norm and condition number of $\mathbf{Q}$ in terms of u_1.
   [*Hint:* In the definition

$$\|\mathbf{Q}\| = \max_{\|\mathbf{v}\|=1} \|\mathbf{Q}\mathbf{v}\| = \max_{\mathbf{v}} \frac{\|\mathbf{Q}\mathbf{v}\|}{\|\mathbf{v}\|},$$

   you may use $\mathbf{v} = [\cos\theta \quad \sin\theta]^T$ without loss of generality, and maximize with respect to θ.]

   (c) At $u_1 = 0.01$ and $u_2 = 0$, find the solution $\mathbf{x}$, norm of $\mathbf{Q}$ and its condition number.

   (d) How does this solution change as a result of very small changes in u_1 and u_2 around these values?

6. For

$$\mathbf{C} = \begin{bmatrix} 2 & 0 & 1 & 3 & 2 \\ 3 & w & 0 & 2 & -1 \\ 3 & -1 & 3 & 7 & 7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix};$$

find the solution of $\mathbf{Cx} = \mathbf{b}$ through Tikhonov regularization and plot it against w for $0 \le w \le 2$. Use different values of ν (say $10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$) to see its effect on the regularized solution.

7. Find the characteristic polynomial of each of the following matrices and **hence** determine the eigenvalues and eigenvectors:

$$\text{(a)} \quad \mathbf{R} = \begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & -2 \\ -4 & 3 & 5 \end{bmatrix}, \qquad \text{(b)} \quad \mathbf{T} = \begin{bmatrix} 0 & 1 & 0 \\ -5 & 5 & -1 \\ -4 & 3 & 0 \end{bmatrix}.$$

8. Find the characteristic polynomial of the following matrix and mention its significance.

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \ddots & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \ddots & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & -a_1 \end{bmatrix}$$

9. Let $\mathbf{A}$ be an $n \times n$ real matrix with real eigenvalues.

   (a) Consider eigenpair $(\lambda_1, \mathbf{q}_1)$ of $\mathbf{A}$, with $\|\mathbf{q}_1\| = 1$. With an $n \times n$ orthogonal matrix $\mathbf{Q}_1 = [\mathbf{q}_1 \;\; \overline{\mathbf{Q}}_1]$, evaluate $\overline{\mathbf{A}}_1 = \mathbf{Q}_1^T \mathbf{A} \mathbf{Q}_1$. Note the sub-diagonal entries in the first column of the result.

   (b) Denote the trailing $(n-1) \times (n-1)$ submatrix of $\overline{\mathbf{A}}_1$ as $\mathbf{A}_1$. Using an eigenpair $(\lambda_2, \mathbf{q}_2)$ (note: $\mathbf{q}_2 \in R^{n-1}$) of $\mathbf{A}_1$, with $\|\mathbf{q}_2\| = 1$, develop an $n \times n$ orthogonal matrix $\mathbf{Q}_2$ which will similarly transform the sub-diagonal part of the second column of $\overline{\mathbf{A}}_1$.

   (c) Precisely state the proposition that the above exercise establishes.

10. For the matrix

$$\mathbf{A} = \begin{bmatrix} 52 & -8 & -16 \\ -26 & 10 & 4 \\ 97 & -19 & -30 \end{bmatrix} \text{ and the starting vector } \mathbf{v}^{(0)} = [0 \;\; 0 \;\; 1]^T,$$

follow the power method iteration

$$\mathbf{w}^{(k)} = \mathbf{A}\mathbf{v}^{(k-1)}, \qquad \mathbf{L} = \begin{bmatrix} \frac{w_1^{(k)}}{v_1^{(k-1)}} & \frac{w_2^{(k)}}{v_2^{(k-1)}} & \frac{w_3^{(k)}}{v_3^{(k-1)}} \end{bmatrix}, \qquad \mathbf{v}^{(k)} = \mathbf{w}^{(k)} / \|\mathbf{w}^{(k)}\|$$

for $k = 1, 2, \cdots$ till $L_1 = L_2 = L_3 = \lambda$ (say), up to one place of decimal. Interpret λ and the current value of $\mathbf{v}^{(k)}$, and also the operation of the entire exercise.