Mathematical Methods in Engineering and Science

IIT Kanpur

Assignment Policy of the Course

- Assignments of this course are of the traditional problem-solving type. Not only the result, but also the approach will be considered for evaluation.
- Submissions are invariably to be in the PDF format. PDF version of scanned copy of neat handwritten work is acceptable. Brevity and to the point solutions will be rewarded. Illegible writing, sloppy work and beating around the bush will be penalized.
- For many segments of the course, a programming background will be useful, though it is not essential.
- For some problems in the assignments, programming will be needed to get the complete solution. But, the programme does not need to be submitted.
  Those students who do not have a programming background may conduct the intial few (2 to 5) steps or iterations manually and indicate through what repititive and automated procedure a programme would need to proceed in order to get the final complete solution. Such solutions will be considered for partial credit.
- There may also be some problems, for which programming may not be needed as such, but may be useful in solving the problem systematically. Students having a programming background are encouraged to take advantage of this.

MMES (2017) Assignment 10

(Full marks = 100)

1. Solve the following differential equations:

   (a) $y'' + xy' - y = e^{3x}$, (b) $x^2y'' + 5xy' + (x+4)y = 0$.

2. (a) Expand $[1 - t(2x - t)]^{-1/2}$ in a binomial series.

   (b) Collect powers of t^k from the above series to establish the *generating function* of Legendre polynomials as

   $$\frac{1}{\sqrt{1 - 2xt + t^2}} = P_0(x) + P_1(x)t + P_2(x)t^2 + P_3(x)t^3 + \cdots .$$

   (c) Differentiate both sides of this equation to derive the recurrence relation

   $$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x).$$

3. For the Chebyshev's equation, $(1 - x^2)y'' - xy' + k^2y = 0$, develop series solutions in the form $\sum_{n=0}^{\infty} a_n x^n$ and explore the possibility of a polynomial solution for integer k.

4. Determine the weight function with respect to which the Laguerre polynomials, i.e. solutions of the Sturm-Liouville problem $xy'' + (1 - x)y' + \nu y = 0,\ 0 \le x < \infty$, are orthogonal.

5. (a) Solve the Chebyshev's equation $(1-x^2)y''-xy'+n^2y = 0$, $n \ge 0$ on the interval $-1 \le x \le 1$ by using the substitution $x = \cos\theta$.

   (b) Concentrate on that family of solutions above for which $y(1) = 1$, and express the solutions for $n = 0, 1, 2, 3, \cdots$ as polynomials $T_n(x)$ in x in terms of a recurrence relation.

   (c) Rearrange the original (Chebyshev) equation as a Sturm-Liouville equation and find the weight function with respect to which the above Chebyshev polynomials $T_n(x)$ are orthogonal to one another.

   (d) Show the orthogonality of the Chebyshev polynomials $T_m(x)$ and $T_n(x)$, $m \ne n$ with respect to the same weight function over $-1 \le x \le 1$ in an independent way, e.g. through direct integration.

   (e) What are the corresponding norms of the Chebyshev polynomials?

   (f) Superimpose $T_0(x), T_1(x), T_2(x), \cdots, T_5(x)$ in the same plot and note your observations.

6. Show that functions $\psi_1(x) = 1$ and $\psi_2(x) = 2x - 1$ over the interval $[0, 1]$ are orthogonal. Develop a function of the form $\psi_3(x) = A + Bx + Cx^2$ to form a three-member orthogonal basis $\{\psi_1, \psi_2, \psi_3\}$ for function approximation in the interval. Verify that an extension of the domain to $[-1, 1]$ through the substitution $t = 2x - 1$ gives Legendre polynomials in t.

7. Determine the Fourier cosine and sine series of the function

$$f(x) = \begin{cases} 1 - \sqrt{1-x^2}, & \text{for} \quad 0 < x < 1, \\ 3 - x, & \text{for} \quad 1 < x < 2, \end{cases}$$

   over $[0, 2]$ and in each case, estimate $f(0)$, $f(1)$ and $f(2)$ from the series.

8. Use Fourier series to find the steady-state current in a single-loop RLC circuit with $R = 100$ ohms, $L = 10$ henrys, $C = 0.01$ farad, and the supply voltage (in volts) as

$$E(t) \;=\; \begin{cases} 100(\pi t + t^2) \text{ for } -\pi \le t \le 0, \\ 100(\pi t - t^2) \text{ for } 0 < t < \pi, \end{cases} \qquad \text{and} \quad E(t + 2\pi) = E(t).$$

9. Approximate e^x in $[-1, 1]$ up to degree three by Taylor's series (about 0), Legendre series and Chebyshev series, and compare their maximum deviations.

10. Find nine Chebyshev accuracy points in $[-5, 5]$ and compare the resulting Chebyshev-Lagrange approximation of $p(x) = \frac{1}{1+x^2}$ with the corresponding (nine-point) Lagrange interpolation with equal spacings.