Unit 11 - Week 9

Course outline

Assignment 9

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-11-18, 23:59 IST.

1) Which of the following topologies are locally compact Hausdorff? *1 point*

- $\mathbb{R}^n$ with standard topology.
- $\mathbb{R} \times \mathbb{R}$, where the first copy of $\mathbb{R}$ has standard topology and the second copy of $\mathbb{R}$ has discrete topology.
- $\mathbb{Q}$ with the relative topology inherited from the standard topology on $\mathbb{R}$.
- $X = \Pi_{\lambda \in \mathbb{R}} X_\lambda$ where each X_λ is a compact Hausdorff space, and X is endowed witvh product topology.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathbb{R}^n$ with standard topology.
$\mathbb{R} \times \mathbb{R}$, where the first copy of $\mathbb{R}$ has standard topology and the second copy of $\mathbb{R}$ has discrete topology.
$X = \Pi_{\lambda \in \mathbb{R}} X_\lambda$ where each X_λ is a compact Hausdorff space, and X is endowed witvh product topology.

2) Determine which of the following are positive linear functionals? *1 point*

- $\Lambda : L^1(\mathbb{R}, m) \to \mathbb{C}$ given by $\Lambda(f) := \int_{\mathbb{R}} f\,dm$
- $\Lambda : C_c(\mathbb{R}^2) \to \mathbb{C}$ given by $\Lambda(f) := m^{(2)}(supp(f))$
- $\Lambda : C_c(\mathbb{R}) \to \mathbb{C}$ given by $\Lambda(f) := f(0)$
- $\Lambda : C_c(\mathbb{R}) \to \mathbb{C}$ given by $\Lambda(f) := \lim_{x \to \infty} f(x)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\Lambda : L^1(\mathbb{R}, m) \to \mathbb{C}$ *given by* $\Lambda(f) := \int_{\mathbb{R}} f\,dm$
$\Lambda : C_c(\mathbb{R}) \to \mathbb{C}$ *given by* $\Lambda(f) := f(0)$
$\Lambda : C_c(\mathbb{R}) \to \mathbb{C}$ *given by* $\Lambda(f) := \lim_{x \to \infty} f(x)$

3) Let X be a locally compact Hausdorff space and $(X, \mathcal{B}, \mu)$ be a measure space. If μ is a Radon measure, which of the following are true? *1 point*

- μ is inner regular for all $\mathcal{B}$ measurable sets in X
- μ is inner regular for all Borel sets in X
- μ is inner regular for all open sets in X
- μ is inner regular for all Borel sets E in X with $\mu(E) < \infty$

No, the answer is incorrect.
Score: 0
Accepted Answers:
μ is inner regular for all open sets in X
μ is inner regular for all Borel sets E in X with $\mu(E) < \infty$

For each $k \in \mathbb{N}$, $A_k \subseteq \mathbb{R}^d$ be the sets,

$$A_k := \{x \in \mathbb{R}^d | k-1 < ||x|| \leq k\}$$

Let $\{x_k\}$ be a sequence of points in $\mathbb{R}^d$ such that $x_k \in A_k$ for each $k \in \mathbb{N}$.

Define $\Lambda : C_c(\mathbb{R}^d) \to \mathbb{C}$

$$\Lambda(f) := \sum_{k=1}^{\infty} \frac{1}{2^k} f(x_k)$$

It can be checked that Λ is a positive linear functional. Answer the next three questions based on this definition.

4) Choose the correct statements: *1 point*

- $|\Lambda(f)|$ is always finite.
- $\exists f \in C_c(\mathbb{R}^d)$ for which $|\Lambda(f)| = +\infty$
- If $f \in C_c(\mathbb{R}^d)$ such that supp$(f) \subseteq A_{n_0}$ for some $n_0 \in \mathbb{N}$, then the following inequality holds:

  $$|\Lambda(f)| \leq (1 - \frac{1}{2^{n_0}})||f||_\infty \quad \text{(Where } ||f||_\infty := sup_{x \in \mathbb{R}^d}|f(x)| \text{)}$$
- Given $N \in \mathbb{N}, \exists f \in C_c(\mathbb{R}^d)$such that $\Lambda(f) = \sum_{k=1}^{N} \frac{1}{2^k}$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$|\Lambda(f)|$ is always finite.
If $f \in C_c(\mathbb{R}^d)$ such that supp$(f) \subseteq A_{n_0}$ for some $n_0 \in \mathbb{N}$, then the following inequality holds:

$$|\Lambda(f)| \leq (1 - \frac{1}{2^{n_0}})||f||_\infty \quad \text{(Where } ||f||_\infty := sup_{x \in \mathbb{R}^d}|f(x)| \text{)}$$

Given $N \in \mathbb{N}, \exists f \in C_c(\mathbb{R}^d)$such that $\Lambda(f) = \sum_{k=1}^{N} \frac{1}{2^k}$

5) If μ_Λ is the measure induced by Λ on $\mathbb{R}^d$, the value of $\mu_\Lambda(\mathbb{R}^d)$ is *1 point*

- $+\infty$
- 1
- 0

No, the answer is incorrect.
Score: 0
Accepted Answers:
1

6) Let $B(0, r)$ be the open Euclidean ball in $\mathbb{R}^d$ of radius $r > 0$. Then *1 point*

- $\mu_\Lambda(B(0,1)) = 0$ or $1/2$
- $\mu_\Lambda(B(0,1)) = 1/2$ or $3/4$
- $\mu_\Lambda(B(0,1)) = 1$ or 0
- $\mu_\Lambda(B(0,1)) = 1$ or $1/2$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mu_\Lambda(B(0,1)) = 0$ or $1/2$

7) Let $G = (\mathbb{R}^*, \times)$ be the multiplicative group of non-zero reals and m_* be the left-invariant Haar measure on G as defined in the lectures *1 point*

- If $0 < a < b, m_*([a,b]) = ln(b/a)$
- If $x > 0$, then $m_*([a+x, b+x]) = m_*([a,b])$
- If $c > 0$ and $0 < a < b$ then $m_*([ca, cb]) = m_*([a,b])$
- m_* is a Radon measure on G.

No, the answer is incorrect.
Score: 0
Accepted Answers:
If $0 < a < b, m_*([a,b]) = ln(b/a)$
If $c > 0$ and $0 < a < b$ then $m_*([ca, cb]) = m_*([a,b])$
m_* is a Radon measure on G.

8) Consider the counting measure μ on $\mathbb{R}$. Then there exists a positive linear functional Λ on $C_c(\mathbb{R})$ such that the measure induced by Λ is μ. *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
False

9) Let X be a locally compact Hausdorff space. Which of the following statements are true? *1 point*

- For any Borel measure μ on X, the functional $\Lambda_\mu : C_c(X) \to \mathbb{C}, f \mapsto \int f\,d\mu$ on $C_c(X)$ is always well defined.
- Λ_μ is well defined if and only if μ is finite on compact subsets of X,where Λ_μ is defined as in (a).
- Suppose μ is a Radon measure on X. If $f \in C_c(X)$ and $f(x) \geq 0 \forall x \in X$, then $f \notin Ker(\Lambda_\mu)$ unless $f \equiv 0$ on X.

No, the answer is incorrect.
Score: 0
Accepted Answers:
Λ_μ is well defined if and only if μ is finite on compact subsets of X,where Λ_μ is defined as in (a).
Suppose μ is a Radon measure on X. If $f \in C_c(X)$ and $f(x) \geq 0 \forall x \in X$, then $f \notin Ker(\Lambda_\mu)$ unless $f \equiv 0$ on X.

10) Let X be a locally compact Hausdorff space. Let μ be a Radon measure on X. *1 point*

- If τ is the measure induced by the positive linear functional $f \mapsto \int f d\mu$, then τ and μ have same null sets.
- Suppose β is a Borel measure on X which is finite on compact sets, then there exists a Radon measure γ on X such that a β-null set is a γ-null net.
- $\mu(U) > 0$ for all open subsets U of X.

No, the answer is incorrect.
Score: 0
Accepted Answers:
If τ is the measure induced by the positive linear functional $f \mapsto \int f d\mu$, then τ and μ have same null sets.
Suppose β is a Borel measure on X which is finite on compact sets, then there exists a Radon measure γ on X such that a β-null set is a γ-null net.

11) Let X be a compact Hausdorff space, and μ be a finite Borel measure on X. Which of the following suggest that ψ is the positive linear functional $\psi : C(X) \to \mathbb{C}, f \mapsto \int f d\mu$ on $C(X)$? *1 point*

- $\psi(f) = \int f d\mu$ for all real valued $f \in C(X)$
- $\psi(f) \leq \int f d\mu$ for all real valued $f \in C(X)$
- $\psi(f) \leq \int f d\mu$ for all non-negative real valued $f \in C(X)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\psi(f) = \int f d\mu$ for all real valued $f \in C(X)$
$\psi(f) \leq \int f d\mu$ for all real valued $f \in C(X)$
$\psi(f) \leq \int f d\mu$ for all non-negative real valued $f \in C(X)$