Unit 9 - Week 7

Assignment 7

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-11-04, 23:59 IST.

Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable. Let $\{\phi_k\}$ be a non decreasing sequence of piecewise constant functions such that $\phi_k\le f\,\forall k\ge 1, |\phi_k(x)|\le \sup_{[a,b]}|f(x)|\forall x\in[a,b]$ and

$$\int_a^b \phi_k(x)dx \to \int_a^b f(x)dx, \quad \text{as } k\to\infty$$

(This sequence exists due to the Riemann-Darboux theorem). Similarly, let $\{\psi_k\}$ be a sequence of non increasing piecewise constant functions such that $f\le\psi_k, |\psi_k(x)|\le \sup_{x\in[a,b]}|f(x)|, \forall x\in[a,b]$ and

$$\int_a^b \psi_k(x)dx \to \int_a^b f(x)dx, \text{ as } k\to\infty$$

Now answer the next two questions.

1) Which of the following are true? *1 point*

- $\phi=\lim_{k\to\infty}\phi_k$ and $\psi=\lim_{k\to\infty}\psi_k$ are measurable functions
- ϕ and ψ defined above are Lebesgue integrable on $[a,b]$
- The Riemann integral of ϕ_k, $\int_a^b \phi_k(x)dx$ is equal to the Lebesgue integral of ϕ_k, $\int_{[a,b]}\phi_k dm$ for each $k\ge 1$
- The Riemann integral of ψ_k, $\int_a^b \psi_k(x)dx$ is NOT equal to the Lebesgue integral of ψ_k, $\int_{[a,b]}\psi_k dm$ for each $k\ge 1$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\phi=\lim_{k\to\infty}\phi_k$ and $\psi=\lim_{k\to\infty}\psi_k$ are measurable functions
ϕ and ψ defined above are Lebesgue integrable on $[a,b]$
The Riemann integral of ϕ_k, $\int_a^b \phi_k(x)dx$ is equal to the Lebesgue integral of ϕ_k, $\int_{[a,b]}\phi_k dm$ for each $k\ge 1$

2) Which of the following are true? *1 point*

- $\lim_{k\to\infty}\int_{[a,b]}\phi_k\,dm=\int_{[a,b]}\phi\,dm$ and $\int_{[a,b]}\psi_k\,dm=\int_{[a,b]}\psi\,dm$
- f may not be a real measurable function
- $f=\phi=\psi$ for x $a.e.$ in $[a,b]$
- The Riemann integral f, $\int_a^b f(x)dx$ is equal to the Lebesgue integral of f, $\int_{[a,b]} f\,dm$.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\lim_{k\to\infty}\int_{[a,b]}\phi_k\,dm=\int_{[a,b]}\phi\,dm$ and $\int_{[a,b]}\psi_k\,dm=\int_{[a,b]}\psi\,dm$
$f=\phi=\psi$ for x $a.e.$ in $[a,b]$
The Riemann integral f, $\int_a^b f(x)dx$ is equal to the Lebesgue integral of f, $\int_{[a,b]} f\,dm$.

3) Let $a\in\mathbb{R}$. Let $f_a:\mathbb{R}\to[0,\infty]$ be defined by $f_a(x)=e^{-x}x^{a-1}\chi_{[0,\infty)}(x)$. Which of the following are true? *1 point*

[Hint: Use the fact that $\int_{[0,\infty)} f_a\,dm=\lim_{n\to\infty}\int_{1/n}^1 f_a(x)dx+\lim_{n\to\infty}\int_1^n f_a(x)dx$, where the integrals under the limits are ordinary Riemann integrals]

- $f_a\in L^1(\mathbb{R},m)$ for $a>0$
- $f_a\in L^1(\mathbb{R},m)$ for $a>-1$
- $f_a\notin L^1(\mathbb{R},m)$ for any $a\in\mathbb{R}$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$f_a\in L^1(\mathbb{R},m)$ for $a>0$

4) Let $d\in\mathbb{N},\sigma\in[0,\infty)$ and $f:\mathbb{R}^d\to[0,\infty)$ be defined as *1 point*

$$f(x):=(1+||x||^\sigma)^{-1}$$

Choose the correct statements from the following.

Hint: To estimate $\int_{\overline{B(0,1)}^c} f\,dm$, Let $A_k=\{x\in\overline{B(0,1)}^c:2^k<||x||\le 2^{k+1}\}, k\ge 0$. Note that $f(x)\le(1+2^{k\sigma})^{-1}$ on A_k. Define $\alpha_k(x)=(1+2^{k\sigma})^{-1}\chi A_k(x), k\ge 0.$ Now $f(x)\le\sum_0^\infty\alpha_k(x)$ and use Tonelli's theorem.

- $f\in L^1(\mathbb{R}^d,m)$ if and only if $\sigma>d$
- $f\in L^1(\mathbb{R}^d,m)$ if and only if $0<\sigma<d$
- $f\in L^1(\mathbb{R}^d,m)$ if $\sigma=d$
- $f\in L^1(\mathbb{R}^d,m)$ if $d/2\le\sigma\le d$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$f\in L^1(\mathbb{R}^d,m)$ if and only if $\sigma>d$

In the following questions 5-8, use the following fact wherever applicable: the improper Riemann integral $\int_0^\infty f(x)dx:=\lim_{R\to\infty}\int_0^R f(x)dx$ coincides with the Lebesgue integral for any unsigned measurable function f, which is continuous $[0,\infty)$

5) Evaluate the following limit. *1 point*

$$\lim_{n\to\infty}\int_0^\infty\left(1+\frac{x}{n}\right)^{-n} cos\left(\frac{x}{n}\right)dx$$

[Hint: Use the inequality: $(1+x/n)^{-n}\le(1+x+x^2/4)^{-1}$ for $x\ge 0$]

- 0
- 1
- ∞

No, the answer is incorrect.
Score: 0
Accepted Answers:
1

6) Evaluate the following limit: *1 point*

$$\lim_{n\to\infty}\int_0^1\frac{ln(1+nx)}{(1+x)^n}dx$$

[Hint: Use the inequality $(1+x)^n\ge 1+nx,$ for $x\ge 0$]

- 0
- 1
- ∞

No, the answer is incorrect.
Score: 0
Accepted Answers:
0

7) Evaluate the following limit: *1 point*

$$\lim_{n\to\infty}\int_0^\infty\int_0^n \sin(x)e^{-tx}dxdt$$

[Hint : Evaluate the inner integral explicitly.]

- 0
- π
- $\pi/2$
- 1

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\pi/2$

8) Evaluate the limit: *0 points*

$$\lim_{n\to\infty}\int_0^\infty n\,sin\left(\frac{x}{n}\right)[x(1+x^2)^{-1}]dx$$

[Hint:Evaluate the limit $\lim_{\epsilon\to 0^+}\frac{sin\,\epsilon}{\epsilon}$]

- π
- $\pi/2$
- 0
- 2π

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\pi/2$

9) Let $f\in L^1(\mathbb{R}^d,m)$. Which of the following are true? *1 point*

- Given $\epsilon>0$, there exists $R>0$ such that $\int_{B(0,R)^c}|f|dm<\epsilon$
- If f_n is a sequence of integrable functions such that $||f_n-f||_{L^1}\to 0$ as $n\to\infty$ then f_n converges to f pointwise almost everywhere.
- For $n\ge 1$ let $g_n=\min\{|f|,n\}$. Then each g_n is integrable and $\lim_{n\to\infty}\int_{\mathbb{R}^d}g_n dm=\int_{\mathbb{R}^d}|f|\,dm$
- Let $\{f_n\}$ be a sequence of integrable functions such that $||f_n-f||_{L^1}\to 0$ as $n\to\infty$. Then given $\epsilon>0$ and $\lambda\ge 1$ there exists a $N\in\mathbb{N}$ such that $m(\{x\in\mathbb{R}^d:|f_n(x)-f(x)|>\lambda\})\le\epsilon\,\forall n\ge N$

No, the answer is incorrect.
Score: 0
Accepted Answers:
Given $\epsilon>0$, there exists $R>0$ such that $\int_{B(0,R)^c}|f|dm<\epsilon$
For $n\ge 1$ let $g_n=\min\{|f|,n\}$. Then each g_n is integrable and $\lim_{n\to\infty}\int_{\mathbb{R}^d}g_n dm=\int_{\mathbb{R}^d}|f|\,dm$
Let $\{f_n\}$ be a sequence of integrable functions such that $||f_n-f||_{L^1}\to 0$ as $n\to\infty$. Then given $\epsilon>0$ and $\lambda\ge 1$ there exists a $N\in\mathbb{N}$ such that $m(\{x\in\mathbb{R}^d:|f_n(x)-f(x)|>\lambda\})\le\epsilon\,\forall n\ge N$

10) Let $f\in L^1(\mathbb{R},m)$. Then, given $\epsilon>0$ there exists $\delta>0$ such that if $A\subset\mathbb{R}^d$ is a measurable subset such that $m(A)<\delta$, then $\int_A|f|dm<\epsilon$ *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

11) Suppose $\{f_n\}$ is a sequence in $L^1(\mathbb{R},m)$ and $f:\mathbb{R}\to\mathbb{C}$ is a measurable fuction such that $\int_{\mathbb{R}}|f_n-f|\to 0$ as $n\to\infty$. Then which of the following are true? *1 point*

- There exists $M>0$ such that $|f_n(x)|\le M\ \forall n\ge 1$ for $a.e.\,x\in\mathbb{R}$
- $f\in L^1(\mathbb{R},m)$
- There exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges point wise a.e. to f.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$f\in L^1(\mathbb{R},m)$
There exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges point wise a.e. to f.

12) Choose the correct statements from the following. *1 point*

- Let $f\in L^1(\mathbb{R}^d,m)$, and let $\epsilon>0$. Then there exists a Lebesgue measurable set $E\subset\mathbb{R}^d, m(E)<\epsilon$ such that the restriction of f to $\mathbb{R}^d\backslash E$ is continuous.
- Let $f\in L^1(\mathbb{R}^d,m)$, and let $\epsilon>0$. Then there exists a Lebesgue measurable set $E\subset\mathbb{R}^d, m(E)<\epsilon$ such that f is continuous on every point in $\mathbb{R}^d\backslash E$, viewed as a function on $\mathbb{R}^d$.
- Let $f\in L^1(\mathbb{R}^d,m)$, then f is continuous on a G_δ subset of $\mathbb{R}^d$

No, the answer is incorrect.
Score: 0
Accepted Answers:
Let $f\in L^1(\mathbb{R}^d,m)$, and let $\epsilon>0$. Then there exists a Lebesgue measurable set $E\subset\mathbb{R}^d, m(E)<\epsilon$ such that the restriction of f to $\mathbb{R}^d\backslash E$ is continuous.

13) Let $(X,\mathcal{B},\mu)$ be a measure space and $f:X\to\mathbb{C}$ be measurable.. Then which of the following are true? *1 point*

- If f_n is a sequence of integrable functions converging uniformly to f, then f is integrable.
- If $\mu(X)<\infty, (a)$ holds.
- If f_n is a sequence of bounded integrable functions converging uniformly to f, then f is integrable.
- Suppose $\mu(X)<\infty$ and the sequence $\{f_n\}$ converges to f pointwise a.e. to f. If $|f_n(x)|\le M$ for $a.e.\,x\in X$ and for every $n\ge 1,$ then f is integrable and $\lim_{n\to\infty}\int_X f_n d\mu=\int_X f d\mu$

No, the answer is incorrect.
Score: 0
Accepted Answers:
If $\mu(X)<\infty, (a)$ holds.
Suppose $\mu(X)<\infty$ and the sequence $\{f_n\}$ converges to f pointwise a.e. to f. If $|f_n(x)|\le M$ for $a.e.\,x\in X$ and for every $n\ge 1,$ then f is integrable and $\lim_{n\to\infty}\int_X f_n d\mu=\int_X f d\mu$

14) Let $(X,\mathcal{B},\mu)$ be a measure space and $f:X\to\mathbb{C}$ be measurable.Then which of the following are true? *1 point*

- If f is bounded a.e., then it is integrable.
- If $\mu(X)<\infty$ then f is integrable.
- If f is bounded a.e. and $\mu(X)<\infty,$ then f is integrable.

No, the answer is incorrect.
Score: 0
Accepted Answers:
If f is bounded a.e. and $\mu(X)<\infty,$ then f is integrable.

15) Which of the following are dense in the Banach space $L^1(\mathbb{R},m)$? *1 point*

- The class of simple measurable functions on $\mathbb{R}$
- The class of step functions on $\mathbb{R}$
- The class of compactly supported continuous functions on $\mathbb{R}$

No, the answer is incorrect.
Score: 0
Accepted Answers:
The class of simple measurable functions on $\mathbb{R}$
The class of step functions on $\mathbb{R}$
The class of compactly supported continuous functions on $\mathbb{R}$