NPTEL » Measure Theory

## Unit 8 - Week 6

Course outline

course work?

Week 0

Week 1

Week 2

Week 3

Week 4

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Week 6

Part 1

Part 2

properties

Measure Theory

Lecture notes

Week 7

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Announcements About the Course Ask a Question Progress Mentor
                                                       Assignment 6
How does an NPTEL online
                                                       The due date for submitting this assignment has passed.
                                                                                                                                                                                                            Due on 2020-10-28, 23:59 IST.
                                                       As per our records you have not submitted this assignment.
                                                          • A statement on a measure space (X, \mathcal{B}, \mu) is said to hold \mu- almost everywhere (or simply almost everywhere when the underlying measure is clear from
                                                              the context) if the statement holds on the complement of a \mu null set. For example 'f: X \to \mathbb{R} is zero \mu-almost everywhere' means that there exists
                                                              A \in \mathcal{B} and \mu(A) = 0 such that f is identically zero on A^c
                                                          • Let (X,\mathcal{B},\mu) be a measure space and f be an unsigned measurable function on X. Then f is said to be integrable if and only if \int_X f d\mu < \infty

    Let (X, B) be a measurable space. Then which of the following statements are true?

                                                                                                                                                                                                                                                      1 point
   Measurable functions:
                                                         If f is (\mathcal{B},\mathcal{B}(\mathbb{R}^m)) measurable, then f is (\mathcal{B},\mathcal{L}(\mathbb{R}^m)) measurable.
    definition and basic properties -
                                                         If X is a totally ordered set with its order topology, then any order preserving f:X	o [0,\infty] is measurable.
   Measurable functions:
    definition and basic properties -
                                                          \mathcal{A} = \{f : X \to \mathbb{C} \text{ is measurable}\} is a \mathbb{C}- vector space.
    Egorov's theorem: abstract
                                                         If f:\mathbb{R} 	o \mathbb{R} is measurable, then it is almost everywhere continuous. i.e, there exists A \in \mathcal{L}(\mathbb{R}) such that \mu(A) = 0 and f is continuous on A^c
                                                       No, the answer is incorrect.

    Lebesgue integral of unsigned

                                                        Score: 0
    simple measurable functions:
    definition and properties
                                                       If X is a totally ordered set with its order topology, then any order preserving f: X \to [0, \infty] is
   Lebesgue integral of unsigned
                                                       \mathcal{A} = \{f: X \to \mathbb{C} \text{ is measurable}\} is a \mathbb{C}-vector space.
    measurable functions:
    motivation, definition and basic
                                                       2) Let (X, \mathcal{B}, \mu) be a measurable space.Let f: X \to \mathbb{R} be measurable. Choose the correct statements:
                                                                                                                                                                                                                                                      1 point
   Fundamental convergence
    theorems in Lebesgue
                                                         If f \ge 0 there exists a sequence (\phi_n) of simple measurable functions such that 0 \le \phi_1 \le \phi_2 \le \cdots \le f and (\phi_n) converges pointwise to f.
    integration: Monotone
    convergence theorem, Tonelli's
    theorem and Fatou's lemma
                                                         If (X, \mathcal{B}, \mu) is \sigma-finite, there exists a sequence of simple measurable functions (\phi_n) such that each (\phi_n) is integrable and (\phi_n) converges to f pointwise.
   Quiz : Assignment 6
                                                         If f is bounded, f \ge 0, there exists a sequence (\phi_n) of simple measurable functions such that 0 \le \phi_1 \le \phi_2 \le \cdots \le f and (\phi_n) converges uniformly to f.
   Week 6 Feedback Form:
                                                         If (X,\mathcal{B},\mu) is \sigma-finite, there exists a sequence of simple measurable functions (\phi_n) such that each (\phi_n) is integrable and (\phi_n) converges to f uniformly.
                                                        No, the answer is incorrect.

    Assignment-6 solutions

                                                        Score: 0
                                                        Accepted Answers:
                                                       If f \geq 0 there exists a sequence (\phi_n) of simple measurable functions such that 0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f
                                                       and (\phi_n) converges pointwise to f .
                                                       If (X, \mathcal{B}, \mu) is \sigma-finite, there exists a sequence of simple measurable functions (\phi_n) such that each
                                                       (\phi_n) is integrable and (\phi_n) converges to f pointwise.
                                                       If f is bounded, f \geq 0, there exists a sequence (\phi_n) of simple measurable functions such that
                                                       0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f and (\phi_n) converges uniformly to f .

 Let (X, B) be a measurable space. Choose the correct statements from the following:

                                                                                                                                                                                                                                                      1 point
                                                         If f,g:X	o\mathbb{C} are measurable and g(x)
eq 0 \forall x\in X , then f/g is measurable.
                                                         If f,g:X\to\mathbb{C} are measurable, then f,g is measurable.
                                                       No, the answer is incorrect.
                                                        Score: 0
                                                        Accepted Answers:
                                                       If f,g:X	o\mathbb{C} are measurable and g(x)\neq 0 \forall x\in X, then f/g is measurable.
                                                       If f,g:X	o\mathbb{C} are measurable , then f.g is measurable.
                                                       4) For a measurable space (X, \mathcal{B}), which of the following are true?
                                                                                                                                                                                                                                                      1 point
                                                         If f_n:X	o\mathbb{R} is measurable orall n\in\mathbb{N} then F:X	o\mathbb{R} defined by F(x)=\limsup_{n	o\infty}f_n(x) is measurable.
                                                         If f_n:X\to\mathbb{R} is measurable and (f_n) converges pointwise to F:X\to\mathbb{R}, then F is measurable.
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       Accepted Answers:
                                                       If f_n:X	o\mathbb{R} is measurable \,\,orall n\in\mathbb{N} then F:X	o\mathbb{R} defined by \,F(x)=\,\, sup_{n	o\infty}f_n(x)\, is
                                                       If f_n:X	o\mathbb{R} is measurable \forall n\in\mathbb{N} then F:X	o\mathbb{R} defined by F(x)= lim \sup_{n	o\infty}f_n(x) is
                                                       If f_n:X\to\mathbb{R} is measurable and (f_n) converges pointwise to F:X\to\mathbb{R}, then F is measurable.
                                                       5) For a measure space (X, \mathcal{B}, \mu) which of the following statements are correct?
                                                                                                                                                                                                                                                     0 points
                                                         If f,g:X	o\mathbb{R} and f=g\,\mu\,a.\,e. then g is measurable.
                                                         If f_n:X	o\mathbb{R} and (f_n) converges to f pointwise \mu a.e,., then f is measurable.
                                                          Both (a) and (b) are true if and only if X is a complete measure space. Recall that a measure space (X, \mathcal{B}, \mu) is called complete if A \in \mathcal{B} is such that
                                                         \mu(A)=0, then the power set of A,\mathcal{P}(A)\subseteq\mathcal{B}.
                                                       No, the answer is incorrect.
                                                       Accepted Answers:
                                                       Both (a) and (b) are true if and only if X is a complete measure space. Recall that a measure space
                                                       (X,\mathcal{B},\mu) is called complete if A\in\mathcal{B} is such that \mu(A)=0, then the power set of A,\mathcal{P}(A)\subseteq\mathcal{B}.
                                                       6) Suppose (X, \mathcal{B}, \mu) is a finite measure space and let f_n: X \to [0, \infty] be measurable for n \in \mathbb{N}, and (f_n) converges pointwise to f, then f converges 1 point
                                                     uniformly almost everywhere to f.
                                                         True

    False

                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       Accepted Answers:
                                                       7) Which of the following are true?
                                                                                                                                                                                                                                                      1 point
                                                         If f: \mathbb{R}^d \to [0, \infty] is an unsigned measurable function and g: \mathbb{R}^d \to \mathbb{C} is complex measurable, then the pointwise product of f and Im(g) is a real measurable
                                                         If f: \mathbb{R}^d \to \mathbb{C} is continuous, then it is measurable.
                                                         If f:\mathbb{R}^d\to\mathbb{R} is continuous, and E\subset\mathbb{R} is Lebesgue measurable, then f^{-1}(E) is Lebesgue measurable.
                                                         The indicator function of a Lebesgue measurable subset of \mathbb{R}^d is an unsigned measurable function.
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       Accepted Answers:
                                                       If f:\mathbb{R}^d	o [0,\infty] is an unsigned measurable function and g:\mathbb{R}^d	o \mathbb{C} is complex measurable, then the
                                                       pointwise product of f and Im(g) is a real measurable function.
                                                       If f: \mathbb{R}^d \to \mathbb{C} is continuous, then it is measurable.
                                                       The indicator function of a Lebesgue measurable subset of \mathbb{R}^d is an unsigned measurable function.
                                                       8) Let X be a set and \mathcal{B}, \mathcal{B}' be two \sigma - algebras on X, let Id: X \to X be the identity map, x \mapsto x. Then which of the following are true?
                                                                                                                                                                                                                                                      1 point
                                                         If \mathcal{B} \subseteq \mathcal{B}' then Id is (\mathcal{B}, \mathcal{B}')- measurable.
                                                         If \mathcal{B}'\subseteq\mathcal{B} then Id is (\mathcal{B},\mathcal{B}')- measurable.
                                                         Id is always (\mathcal{B}, \mathcal{P}(X))- measurable.
                                                         Id is always (\mathcal{P}(X), \mathcal{P}(X))- measurable.
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       Accepted Answers:
                                                       If \mathcal{B}' \subseteq \mathcal{B} then Id is (\mathcal{B}, \mathcal{B}')- measurable.
                                                       Id is always (\mathcal{P}(X), \mathcal{P}(X)) - measurable.
                                                       9) Let f: \mathbb{R}^d \to \mathbb{R} and g: \mathbb{R}^d \to \mathbb{R}^d be two functions. Then which of the following are true?
                                                                                                                                                                                                                                                      1 point
                                                         If both f and g are continuous, f \circ g is measurable.
                                                         If f is real measurable and g is continuous, then f \circ g is measurable.
                                                        If f is continuous and g is (\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))- measurable, then f \circ g is measurable.
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       Accepted Answers:
                                                       If both f and g are continuous, f \circ g is measurable.
                                                       If f is continuous and g is (\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^d))- measurable, then f\circ g is measurable.
                                                       10) Let (f_n) be a sequence of real measurable functions on \mathbb{R}^d. Let \lambda \in (0, \infty). Which of the following are true?
                                                                                                                                                                                                                                                      1 point
                                                          ig\{x\in\mathbb{R}^d: \sup_{n\in\mathbb{N}}f_n(x)>\lambdaig\}=igcup_{n\in\mathbb{N}}ig\{x\in\mathbb{R}^d: f_n(x)>\lambdaig\}
                                                          \left\{x\in\mathbb{R}^d: \sup_{n\in\mathbb{N}}f_n(x)>\lambda
ight\}=igcap_{n\in\mathbb{N}}\left\{x\in\mathbb{R}^d: f_n(x)>\lambda
ight\}
                                                          \left\{x\in\mathbb{R}^d:\inf_{n\in\mathbb{N}}\sup_{k\geq n}f_k(x)>\lambda
ight\}=igcup_{M=1}^\inftyigcap_{N=1}^\infty\left\{x\in\mathbb{R}^d:\sup_{k\geq N}f_k(x)>\lambda+1/M
ight\}
                                                         \left\{x\in\mathbb{R}^d:\inf_{n\in\mathbb{N}}\sup_{k\geq n}f_k(x)>\lambda
ight\}=igcup_{M=1}^\inftyigcap_{N=1}^\inftyigcup_{k=N}^\infty\left\{x\in\mathbb{R}^d:f_k(x)>\lambda+1/M
ight\}
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                        Accepted Answers:
                                                       ig\{x\in\mathbb{R}^d: sup _{n\in\mathbb{N}}f_n(x)>\lambdaig\}=igcup_{n\in\mathbb{N}}ig\{x\in\mathbb{R}^d:f_n(x)>\lambdaig\}
                                                       \left\{x\in\mathbb{R}^d: \mathit{inf}_{n\in\mathbb{N}}\mathit{sup}_{k\geq n} f_k(x)>\lambda
ight\} = igcup_{M=1}^\inftyigcap_{N=1}^\infty\left\{x\in\mathbb{R}^d: \mathit{sup}_{k\geq N} f_k(x)>\lambda+1/M
ight\}
                                                       ig\{x\in\mathbb{R}^d: \mathit{inf}_{n\in\mathbb{N}}\mathit{sup}_{k\geq n}\ f_k(x)>\lambdaig\}=igcup_{M=1}^\inftyigcap_{N=1}^\inftyigcup_{k=N}^\inftyig\{x\in\mathbb{R}^d: f_k(x)>\lambda+1/Mig\}
                                                       11) Let (X, \mathcal{B}, \mu) be a measure space such that \mu(X) < \infty. Suppose (f_n) is a sequence of functions, f_n : X \to \mathbb{C} is a sequence of measurable
                                                                                                                                                                                                                                                      1 point
                                                     functions such that (f_n) converges pointwise to f: X \to \mathbb{C}, then which of the following are true?
                                                         If X is a topological space and f_n is continuous for each n\geq 1 then given \epsilon>0 there exists a measurable subset A_\epsilon\subseteq X such that \mu(A_\epsilon)\leq \epsilon and f is
                                                          continuous when restricted toA_{\epsilon}^{c}
                                                         If each f_n is integrable, then f is integrable.
                                                         There exists a measurable subset A \subseteq X such that \mu(A) = 0 and f is continuous when restricted to A^c
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       If X is a topological space and f_n is continuous for each n \geq 1 then given \epsilon > 0 there exists a
                                                       measurable subset A_\epsilon \subseteq X such that \mu(A_\epsilon) \le \epsilon ) and f is continuous when restricted to A_\epsilon^c
                                                       12) Let s: \mathbb{R}^d \to [0, \infty) be a simple measurable function. Which of the following are true?
                                                                                                                                                                                                                                                      1 point
                                                         \int_{\mathbb{R}^d} s \, dm = \sup ig\{ \int_{\mathbb{R}^d} \phi \, dm : \phi \leq s, \; \phi \; 	ext{ is simple measurable } ig\}
                                                         Let r:\mathbb{R}^d	o [0,\infty) be another simple measurable function and \alpha,\beta\in [0,\infty), then \int_{\mathbb{R}^d}(\alpha s+\beta r)dm=\alpha\int_{\mathbb{R}^d}s\ dm+\beta\int_{\mathbb{R}^d}r\ dm
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                        \int_{\mathbb{R}^d} \, s \, dm =supig\{ \int_{\mathbb{R}^d} \, \phi \, dm : \phi \leq s, \; \phi \; is simple measurable ig\}
                                                            \int_{\mathbb{R}^d} s \, dm =inf\Big\{ \int_{\mathbb{R}^d} \psi \, dm : s \leq \psi, \; \psi \; is simple measurable \Big\}
                                                       Let r:\mathbb{R}^d	o [0,\infty) be another simple measurable function and lpha,eta\in [0,\infty) , then
                                                       \int_{\mathbb{R}^d} (lpha s + eta r) dm = lpha \int_{\mathbb{R}^d} s \, dm + eta \int_{\mathbb{R}^d} r \, dm
                                                       13) Let (f_n) be a sequence of unsigned measurable function on a measure space (X, \mathcal{B}, \mu)
                                                                                                                                                                                                                                                      1 point
                                                         _{n\to\infty}\int_X f_n d\mu
                                                         _{n
ightarrow\infty}f_{n})d\mu=limsup\int_{X}(lim sup
                                                          \int_X (\liminf_{n 	o \infty} f_n) d\mu = \lim_{n 	o \infty} \int_X \left(\inf_{k \geq n} f_k \right) d\mu
                                                          \int_X (\limsup_{n 	o \infty} f_n) d\mu = \lim_{n 	o \infty} \int_X (\sup_{k \geq n} f_k) \ d\mu
                                                         \int_X (\liminf_{n \to \infty} f_n) d\mu = \liminf_{n \to \infty} \int_X f_n d\mu
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       Accepted Answers:
                                                       \int_X (lim inf_{n	o\infty}f_n)d\mu= lim_{n	o\infty}\int_X (inf_{k\geq n}f_k) d\mu
                                                       14) Suppose (X, \mathcal{B}, \mu) is a measure space. Suppose f: X \to \mathbb{R} is measurable. Which of the following are true?
                                                                                                                                                                                                                                                      1 point
                                                         If f is non-negative and \int_X \ f d\mu = 0 then f \equiv 0
                                                         If f_n:X	o [0,\infty] is measurable orall n\geq 0, then \int_X (\sum_{n=0}^\infty f_n) d\mu = \sum_{n=0}^\infty \int_X f_n \ d\mu
                                                         Suppose X=A\cup B, A,B\in\mathcal{B} with A,B disjoint. Let \mathcal{C}_A=\left\{W\cap A:W\in\mathcal{B}\right\}. Similarly define \mathcal{C}_B . Let \mu_A=\mu|_A, similarly define \mu_B . Now consider the
                                                         measure spaces (A, \mathcal{C}_A, \mu_A) and (B, \mathcal{C}_B, \mu_B). Then g: X \to \mathbb{C} is measurable if and only if g|_A and g|_B are measurable.
                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       If f_n: X 	o [0,\infty] is measurable orall n \geq 0, then \int_X (\sum_{n=0}^\infty f_n) d\mu = \sum_{n=0}^\infty \int_X f_n \ d\mu
                                                       Suppose X=A\cup B, A,B\in \mathcal{B} with A,B disjoint. Let \mathcal{C}_A=ig\{W\cap A:W\in \mathcal{B}ig\} . Similarly define \mathcal{C}_B
                                                       .Let \mu_A=\mu|_A, similarly define \mu_B . Now consider the measure spaces (A,\mathcal{C}_A,\mu_A) and (B,\mathcal{C}_B,\mu_B)
                                                       Then g:X	o\mathbb{C} is measurable if and only if g|_A and g|_B are measurable.
                                                       15) Suppose (X, \mathcal{B}, \mu) is a measure space. Suppose (f_n) is a sequence of unsigned integrable functions converging pointwise to f. If f_n \geq f_{n+1} \forall n \geq 1, 1 point
                                                    then \int_X f \, d\mu = \lim_{n 	o \infty} \int_X f_n \, d\mu
                                                          True

    False

                                                       No, the answer is incorrect.
                                                       Score: 0
                                                       Accepted Answers:
                                                       False
                                                       16) Suppose (X, \mathcal{B}, \mu) is a measure space. Suppose f: X \to [0, \infty] are integrable. Then
                                                                                                                                                                                                                                                      1 point
                                                         f^2 is integrable.
```

Let  $A=ig\{x\in X: f(x)=\inftyig\}.$  Then  $\mu(A)=0$ 

Let  $A=ig\{x\in X: f(x)=\inftyig\}$  . Then  $\mu(A)=0$ 

If  $\mu(A)=0$  for some  $A\in\mathcal{B}$  , then ~eta(A)=0

 $\beta(X) < \infty$  if and only if f is integrable.

If  $\mu(A)=0$  for some  $A\in\mathcal{B}$  , then  $\,eta(A)=0$ 

 $\beta(X) < \infty$  if and only if f is integrable.

 $\mu(A)=0$  for some  $A\in\mathcal{B}$  if and only if  $\beta(A)=0$ 

17) Suppose  $(X, \mathcal{B}, \mu)$  is a measure space. Suppose  $f: X \to [0, \infty]$  is measurable. Define  $\beta: \mathcal{B} \to [0, \infty]$  be defined by  $\beta(A) = \int_A f d\mu$ .

1 point

No, the answer is incorrect.

 $\beta$  is a measure on  $\mathcal{B}$ .

No, the answer is incorrect.

Accepted Answers:  $\beta$  is a measure on  $\mathcal{B}$ .

Score: 0

Accepted Answers: