Unit 6 - Week 4

Assignment 4

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-10-14, 23:59 IST.

Notation:
We use the following notations. The Lebesgue outer measure of a set $E \subseteq \mathbb{R}^d$ is denoted as $m^*(E)$. If $m^*(E) = 0$, E will be called a Lebesgue null set.

$Int(E)$ will denote the interior of a set E and $\overline{E}$ will denote its closure. For a set $A \subseteq \mathbb{R}^d$ we denote by χ_A its characteristic/indicator function.

1) Which of the following are true for a Lebesgue measurable subset $E \subset \mathbb{R}^d$? *1 point*

- $m^*(E) = \sup\{m^*(U) : U \subseteq E$ is open$\}$
- $m^*(E) = \sup\{m^*(A) : A \subseteq E$ is Jordan measurable$\}$
- $m^*(E) = \sup\{m^*(K) : K \subseteq E$ is compact$\}$
- $m^*(E) = \sup\{m^*(F) : F \subseteq E$ is closed$\}$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$m^*(E) = \sup\{m^*(K) : K \subseteq E$ is compact$\}$
$m^*(E) = \sup\{m^*(F) : F \subseteq E$ is closed$\}$

2) Let E, F be bounded, disjoint subsets of $\mathbb{R}^d$. Which of the following conditions imply that $m^*(E \cup F) = m^*(E) + m^*(F)$? *1 point*

- Either E or F is Lebesgue measurable.
- Both E and F are closed.
- If E and F are separated, i.e. $d(E, F) > 0$, where $d(E, F) = \inf\{d(x, y) | x \in E, y \in F\}$.
- Both E and F are arbitrary (but still bounded and disjoint) subsets of $\mathbb{R}^d$.

No, the answer is incorrect.
Score: 0
Accepted Answers:
Either E or F is Lebesgue measurable.
Both E and F are closed.
If E and F are separated, i.e. $d(E, F) > 0$, where $d(E, F) = \inf\{d(x, y) | x \in E, y \in F\}$.

3) Which of the following are true for a Lebesgue measurable subset $E \subseteq \mathbb{R}^d$? *1 point*

- E may contain a Lebesgue non-measurable subset.
- If E is a compact subset of $\mathbb{R}^d$ then every subset of E is Lebesgue measurable.
- If E is a box in $\mathbb{R}^d$, then all its subsets are Lebesgue measurable.

No, the answer is incorrect.
Score: 0
Accepted Answers:
E may contain a Lebesgue non-measurable subset.

4) Recall that a countable intersection of open sets is called a G_δ set and a countable union of closed sets is called an F_σ set. Which of the following implies that a subset $E \subseteq \mathbb{R}^2$ is Lebesgue measurable? *1 point*

- There exists G_δ set $G \supseteq E$ satisfying $m^*(G \backslash E) = 0$
- For every $\epsilon > 0$ there is a closed set $K \subseteq E$ such that $m^*(E \backslash K) < \epsilon$
- There exists a G_δ set G and a F_σ set F such that $F \subseteq E \subseteq G$ such that $m^*(G \backslash F) = 0$

No, the answer is incorrect.
Score: 0
Accepted Answers:
There exists G_δ set $G \supseteq E$ satisfying $m^*(G \backslash E) = 0$
For every $\epsilon > 0$ there is a closed set $K \subseteq E$ such that $m^*(E \backslash K) < \epsilon$
There exists a G_δ set G and a F_σ set F such that $F \subseteq E \subseteq G$ such that $m^*(G \backslash F) = 0$

5) There exists a nowhere dense Lebesgue measurable subset of [0, 1] of positive Lebesgue measure. Recall that $A \subset \mathbb{R}^d$ is called nowhere dense if Int $(\bar{A}) = \phi$. *1 point*
(Hint: Consider generalized Cantor sets)

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

6) Choose correct statements from the following: *1 point*

- There exists a Lebesgue non measurable subset of $\mathbb{R}$ of infinite Lebesgue outer measure.
- Given $\epsilon > 0$, there exists a bounded Lebesgue non-measurable subset of $\mathbb{R}$ of Lebesgue outer measure $< \epsilon$
- Given $M > 0$, there exists a Lebesgue non-measurable subset of $\mathbb{R}$ of Lebesgue outer measure $> M$
- Given $M > 0$, there exists a Lebesgue non-measurable subset of $\mathbb{R}$ of Lebesgue outer measure = M . [Hint: For any $A \subseteq \mathbb{R}^d$, of finite Lebesgue outer measure, for $\lambda > 0$ compute the Lebesgue outer measure of the set $\lambda A := \{\lambda x : x \in A\}$.]

No, the answer is incorrect.
Score: 0
Accepted Answers:
There exists a Lebesgue non measurable subset of $\mathbb{R}$ of infinite Lebesgue outer measure.
Given $\epsilon > 0$, there exists a bounded Lebesgue non-measurable subset of $\mathbb{R}$ of Lebesgue outer measure $< \epsilon$
Given $M > 0$, there exists a Lebesgue non-measurable subset of $\mathbb{R}$ of Lebesgue outer measure $> M$
Given $M > 0$, there exists a Lebesgue non-measurable subset of $\mathbb{R}$ of Lebesgue outer measure = M . [Hint: For any $A \subseteq \mathbb{R}^d$, of finite Lebesgue outer measure, for $\lambda > 0$ compute the Lebesgue outer measure of the set $\lambda A := \{\lambda x : x \in A\}$.]

7) Let $E \subseteq \mathbb{R}$. Then there exists a Lebesgue measurable subset G such that $m(G) = m^*(E)$ *0 points*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

8) Which of the following is true for a Vitali set $V \subset [0, 1]$? *1 point*

- There exists a rational $q \in \mathbb{Q}$ such that $V + q$ is Lebesgue measurable.
- $m^*(V) + m^*([0, 1] \setminus V) = 1$
- $[0, 1] \backslash V$ is Lebesgue non-measurable.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$[0, 1] \backslash V$ is Lebesgue non-measurable.

9) Let $A \subseteq \mathbb{R}$. Then $A \times \{0\}$ is Lebesgue measurable in $\mathbb{R}^2$ *0 points*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

10) Let $\{E_n\}_{n \geq 1}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}^d$ and let $E = \bigcup_{n=1}^{\infty} E_n$. Which of the following are true? *1 point*

- $m(E) = \lim\limits_{n \to \infty} m(\bigcup\limits_{i=1}^{n} E_i)$
- If $m(E) < \infty$ then $m(\limsup\limits_{n \to \infty} E_n) = \limsup\limits_{n \to \infty} m(\bigcup_{k=n}^{\infty} E_k)$
- $m(\liminf\limits_{n \to \infty} E_n) = \liminf\limits_{n \to \infty} m(\bigcap_{k=n}^{\infty} E_k)$
- $m(\liminf\limits_{n \to \infty} E_n) = \liminf\limits_{n \to \infty} m(E_n)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$m(E) = \lim\limits_{n \to \infty} m(\bigcup\limits_{i=1}^{n} E_i)$
If $m(E) < \infty$ *then* $m(\limsup\limits_{n \to \infty} E_n) = \limsup\limits_{n \to \infty} m(\bigcup_{k=n}^{\infty} E_k)$
$m(\liminf\limits_{n \to \infty} E_n) = \liminf\limits_{n \to \infty} m(\bigcap_{k=n}^{\infty} E_k)$

11) Let $\{E_n\}_{n \geq 1}$ be a sequence of Lebesgue measurable subsets of a Euclidean ball B (of radius 1) in $\mathbb{R}^d$ and suppose that there exists a set E such that $\chi_{E_n} \to \chi_E$ pointwise as $n \to \infty$. Which of the following are true? *1 point*

- $m^*(E) = \lim\limits_{n \to \infty} m(E_n)$.
- E is Lebesgue measurable.
- There exists a point $x_0 \in \mathbb{R}^d$ such that $\lim \inf_{n \to \infty} \chi_{E_n}(x_0) \neq \chi_E(x_0)$.
- $E = \lim \sup_{n \to \infty} E_n$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$m^*(E) = \lim\limits_{n \to \infty} m(E_n)$.
E is Lebesgue measurable.
$E = \lim \sup_{n \to \infty} E_n$

12) Consider for $d \geq 1$, d distinct sequences of positive real numbers $\{a_{n,i}\}_{n \in \mathbb{N}}, 1 \leq i \leq d$ such that $\Sigma_{n=1}^{\infty} a_{n,i} < \infty$ for each i, Consider the box $A_n = [0, (a_{n,1})^{1/d}] \times [0, (a_{n,2})^{1/d}] \times \cdots \times [0, (a_{n,d})^{1/d}] \subset \mathbb{R}^d$. Then $m(\lim \sup_{n \to \infty} A_n) =$ ____________.(Hint: Use the A.M.-G.M. inequality) *1 point*

- 0
- ∞
- $\prod_{i=1}^{d} (\sum_{n=1}^{\infty} a_{n,i})^{1/d}$

No, the answer is incorrect.
Score: 0
Accepted Answers:
0

13) Which of the following options imply that a Lebesgue measurable set $E \subset \mathbb{R}^d$ has finite measure? *0 points*

- There exists a compact set F such that $m(E \,\Delta\, F) < \infty$
- $E = \bigcap_{n \geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m(E_n) < \infty$ for some $n \in \mathbb{N}$.
- $E = \bigcup_{n \geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m(E_n) \leq a_n$ for some sequence $\{a_n\}$ such that $\lim_{n \to \infty} a_n = 0$.
- $E = \bigcup_{n \geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m(E_n) < a_n$ for some sequence $\{a_n\}$ such that $\Sigma a_n < \infty$.

No, the answer is incorrect.
Score: 0
Accepted Answers:
There exists a compact set F such that $m(E \,\Delta\, F) < \infty$
$E = \bigcap_{n \geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m(E_n) < \infty$ for some $n \in \mathbb{N}$.
$E = \bigcup_{n \geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m(E_n) < a_n$ for some sequence $\{a_n\}$ such that $\Sigma a_n < \infty$.

14) Let $E \subset \mathbb{R}^d$ be Lebesgue measurable. Then $m(Int(E)) = m(E) = m(\overline{E})$. *0 points*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
False

15) If $E \subset \mathbb{R}$ is Lebesgue measurable, then its boundary ∂E is Lebesgue measurable and $m(\partial E) = 0$ *0 points*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
False

16) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then which of the following is true? *0 points*

- $G_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is Lebesgue measurable in $\mathbb{R}^2$ and $m(G_f) = 0$
- $A_f := \{(x, y) : 0 < y < f(x)\}$ is Lebesgue measurable
- If $E \subset \mathbb{R}$ has finite Lebesgue outer measure, then $f(E)$ also has finite Lebesgue outer measure.
- If f is a homeomorphism then option (c) is true.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$G_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is Lebesgue measurable in $\mathbb{R}^2$ and $m(G_f) = 0$
$A_f := \{(x, y) : 0 < y < f(x)\}$ is Lebesgue measurable

17) Let C be the middle-third Cantor set.'It can be proven that there exists a continuous surjection $g : [0, 1] \to [0, 1]$ whose restriction to C is also surjective.. Based on this fact, conclude which of the following statements are true. *0 points*

- If $E \subset \mathbb{R}$ is a Lebesgue null set and $f : E \to \mathbb{R}$ is continuous (E having subspace topology from $\mathbb{R}$), then $f(D)$ need not be a Lebesgue null set for each $D \subset E$.
- If $E \subset \mathbb{R}$ is a Lebesgue null set and $f : E \to \mathbb{R}$ is continuous (E having subspace topology from $\mathbb{R}$), $f(E)$ always has finite Lebesgue outer measure.
- Let $E, F \subset \mathbb{R}$, if $f : E \to F$ is uniformly continuous, then f preserves Lebesgue outer measure.

No, the answer is incorrect.
Score: 0
Accepted Answers:
If $E \subset \mathbb{R}$ is a Lebesgue null set and $f : E \to \mathbb{R}$ is continuous (E having subspace topology from $\mathbb{R}$), then $f(D)$ need not be a Lebesgue null set for each $D \subset E$.