Unit 5 - Week 3

Assignment 3

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-10-07, 23:59 IST.

Notation: We use the notation $m_J(.)$ (respectively $m^J(.)$) for the inner Jordan measure (respectively the outer Jordan measure). If $E \subset \mathbb{R}^d$ is elementary or Jordan measurable, then the elementary or Jordan measure of E is denoted by $m(E)$. The Lebesgue outer measure of a set $E \subseteq \mathbb{R}^d$ is denoted as $m^*(E)$. If $m^*(E) = 0$, E will be called a Lebesgue null set.

1) Let $E \subset \mathbb{R}^d$ be bounded. Which of the following statements are true? *1 point*

- If E is Jordan measurable, then $m^*(E) = m(E)$
- $m_J(E) \leq m^*(E)$
- $m_J(E) < m^*(E)$
- If E is open, then $m^*(E) = m_J(E)$

No, the answer is incorrect.
Score: 0

Accepted Answers:
If E is Jordan measurable, then $m^*(E) = m(E)$
$m_J(E) \leq m^*(E)$
If E is open, then $m^*(E) = m_J(E)$

2) Let $E \subset \mathbb{R}^d$. Which of the following statements are true? *1 point*

- Every open set is Lebesgue measurable
- Every closed set is Lebesgue measurable
- If E is Jordan measurable, then E is Lebesgue measurable.
- If E is Lebesgue measurable, then E is Jordan measurable.

No, the answer is incorrect.
Score: 0

Accepted Answers:
Every open set is Lebesgue measurable
Every closed set is Lebesgue measurable
If E is Jordan measurable, then E is Lebesgue measurable.

3) For a sequence (a_n) of real numbers which of the following are true? *0 points*

- If $E_n = (-\infty, a_n), n \geq 1$ then $\limsup_{n\to\infty} E_n = (-\infty, \limsup a_n)$
- If $E_n = (a_n, \infty), n \geq 1$ then $\liminf_{n\to\infty} E_n = (\limsup\ a_n, \infty)$
- If $E_n = (\min\{a_n, a_{n+1}\}, \max\{a_n, a_{n+1}\}), n \geq 1$ then $\limsup_{n\to\infty} E_n$ is always non-empty.
- If $E_n = (\min\{a_n, a_{n+1}\}, \max\{a_n, a_{n+1}\})$, and the sequence $\{a_n\}$ converges, then $\limsup_{n\to\infty} E_n = \{\lim_{n\to\infty} a_n\}$

No, the answer is incorrect.
Score: 0

Accepted Answers:
If $E_n = (-\infty, a_n), n \geq 1$ then $\limsup_{n\to\infty} E_n = (-\infty, \limsup a_n)$
If $E_n = (a_n, \infty), n \geq 1$ then $\liminf_{n\to\infty} E_n = (\limsup\ a_n, \infty)$
If $E_n = (\min\{a_n, a_{n+1}\}, \max\{a_n, a_{n+1}\})$, and the sequence $\{a_n\}$ converges, then
$\limsup_{n\to\infty} E_n = \{\lim_{n\to\infty} a_n\}$

4) Let $0 < a \leq 1/3$ and let C_a denote modified Cantor set corresponding to a. Which of the following are true? *1 point*

- C_a is Jordan measurable if and only if $a = 1/3$
- C_a is Lebesgue measurable $\forall a \in (0, 1/3]$
- $[0,1] \backslash C_a$ is Jordan measurable for some a such that $0 < a < 1/3$
- Let $\{a_n\}$ be a sequence of real numbers such that $0 < a_n < 1/3,$ then $\bigcap_{n\geq 1} C_{a_n}$ is Lebesgue measurable.

No, the answer is incorrect.
Score: 0

Accepted Answers:
C_a is Jordan measurable if and only if $a = 1/3$
C_a is Lebesgue measurable $\forall a \in (0, 1/3]$
Let $\{a_n\}$ be a sequence of real numbers such that $0 < a_n < 1/3,$ then $\bigcap_{n\geq 1} C_{a_n}$ is Lebesgue measurable.

5) Let $E \subseteq \mathbb{R}^d$. Choose the statement which is NOT equivalent to other statements: *1 point*

- Given $\epsilon > 0$ there exists an open set U such that $m^*(U \Delta E) \leq \epsilon$
- Given $\epsilon > 0$ there exists an open set U and a closed set F such that $F \subseteq E \subseteq U$ and $m^*(U \backslash F) \leq \epsilon$
- Given $\epsilon > 0$ there exists an elementary subset $F \subseteq E$ such that $m^*(E \backslash F) \leq \epsilon$

No, the answer is incorrect.
Score: 0

Accepted Answers:
Given $\epsilon > 0$ there exists an elementary subset $F \subseteq E$ such that $m^*(E \backslash F) \leq \epsilon$

6) Let $\{E_n\}$ be a collection of bounded Lebesgue measurable subsets of $\mathbb{R}^d$ and let $E = \bigcup_{n\geq 1} E_n$. Which of the following are true, assuming that the inequalities are considered in the extended non-negative reals $[0, +\infty]$? *1 point*

- Given $\epsilon > 0$, there exists a sequence of open sets $\{U_n\}$ such that $E_n \subseteq U_n$ and $m^*(U_n \backslash E_n) \leq \epsilon/2^n$
- There exists a sequence of compact sets $\{K_n\}$ such that $K_n \subseteq E_n$ and $\lim_{n\to\infty} m^*(E_n \backslash K_n) = 0$
- The following inequality may hold: $m^*(E) < m^*(\limsup_{n\to\infty} E_n)$
- $\lim_{n\to\infty} m^*(E \backslash E_n) = 0$

No, the answer is incorrect.
Score: 0

Accepted Answers:
Given $\epsilon > 0$, there exists a sequence of open sets $\{U_n\}$ such that $E_n \subseteq U_n$ and $m^*(U_n \backslash E_n) \leq \epsilon/2^n$
There exists a sequence of compact sets $\{K_n\}$ such that $K_n \subseteq E_n$ and $\lim_{n\to\infty} m^*(E_n \backslash K_n) = 0$

7) Which of the following statements are true? *1 point*

- Every subset of the middle-thirds Cantor set is Lebesgue measurable.
- Let $E \subset \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}^d$ be a homeomorphism. Then $m^*(f(E)) = m^*(E)$
- Given $\epsilon > 0$ there exists an open dense subset of $\mathbb{R}$ of Lebesgue outer measure less than ϵ.

No, the answer is incorrect.
Score: 0

Accepted Answers:
Every subset of the middle-thirds Cantor set is Lebesgue measurable.
Given $\epsilon > 0$ there exists an open dense subset of $\mathbb{R}$ of Lebesgue outer measure less than ϵ.

8) Which of the following statements are true? *1 point*

- If $T : \mathbb{R}^d \to \mathbb{R}^d$ is linear E is Lebesgue measurable, then $m^*(T(E)) = m^*(E)$ is not always true.
- Lebesgue null sets are always bounded.
- If $A \subset \mathbb{R}$ has positive Lebesgue outer measure, then it contains a non-empty open set.

No, the answer is incorrect.
Score: 0

Accepted Answers:
If $T : \mathbb{R}^d \to \mathbb{R}^d$ is linear E is Lebesgue measurable, then $m^*(T(E)) = m^*(E)$ is not always true.

9) Let $A \subset \mathbb{R}$. Then $A \times \{0\}$ is Lebesgue measurable in $\mathbb{R}^2$ *1 point*

- True
- False

No, the answer is incorrect.
Score: 0

Accepted Answers:
True

10) Which of the following options imply that a Lebesgue measurable set $E \subset \mathbb{R}^d$ has finite Lebesgue outer measure? *1 point*

- There exists a compact set F such that $m^*(E \Delta F) < \infty$
- $E = \bigcap_{n\geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m^*(E_n) < \infty$ for some $n \in \mathbb{N}$
- $E = \bigcup_{n\geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m^*(E_n) < a_n$ for some sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 0.$
- $E = \bigcup_{n\geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m^*(E_n) < a_n$ for some sequence $\{a_n\}$ such that $\sum a_n < \infty$

No, the answer is incorrect.
Score: 0

Accepted Answers:
There exists a compact set F such that $m^*(E \Delta F) < \infty$
$E = \bigcap_{n\geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m^*(E_n) < \infty$ for some $n \in \mathbb{N}$
$E = \bigcup_{n\geq 1} E_n$ where $\{E_n\}$ is a sequence of Lebesgue measurable sets such that $m^*(E_n) < a_n$ for some sequence $\{a_n\}$ such that $\sum a_n < \infty$

11) Let $E \subset \mathbb{R}^d$ be Lebesgue measurable. Then $m^*(Int(E)) = m^*(E) = m^*(\overline{E})$, where $Int(E)$ is the interior of E and $\overline{E}$ is the closure of E. *1 point*

- True
- False

No, the answer is incorrect.
Score: 0

Accepted Answers:
False

12) If $E \subset \mathbb{R}$ is Lebesgue measurable, then its boundary ∂E is Lebesgue measurable and $m^*(\partial E) = 0$ *1 point*

- True
- False

No, the answer is incorrect.
Score: 0

Accepted Answers:
False

13) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then which of the following is true? *1 point*

- $G_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is Lebesgue measurable in $\mathbb{R}^2$ and $m(G_f) = 0$
- $A_f := \{(x, y) : 0 < y < f(x)\}$ is Lebesgue measurable
- If $E \subset \mathbb{R}$ has finite Lebesgue outer measure, then $f(E)$ also has finite Lebesgue outer measure.
- If E has positive Lebesgue outer measure, then $f(E)$ has positive Lebesgue outer measure.

No, the answer is incorrect.
Score: 0

Accepted Answers:
$G_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is Lebesgue measurable in $\mathbb{R}^2$ and $m(G_f) = 0$
$A_f := \{(x, y) : 0 < y < f(x)\}$ is Lebesgue measurable

14) Let C be the middle-third Cantor set. It can be proved that there exists a continuous surjection $g : C \to [0,1]$. Based on this fact, conclude which of the following statements are true? *1 point*

- If $E \subset \mathbb{R}^d$ is a Lebesgue null set and $f : E \to \mathbb{R}$ is continuous (E having subspace topology from $\mathbb{R}$), $f(D)$ need not be a Lebesgue null set for each $D \subset E$.
- If $E \subset \mathbb{R}^d$ is a Lebesgue null set and $f : E \to \mathbb{R}$ is continuous (E having subspace topology from $\mathbb{R}$), $f(E)$ always has finite Lebesgue outer measure.
- Let $E, F \subset \mathbb{R}^d$, if $f : E \to F$ is uniformly continuous, then f preserves Lebesgue outer measure.

No, the answer is incorrect.
Score: 0

Accepted Answers:
If $E \subset \mathbb{R}^d$ is a Lebesgue null set and $f : E \to \mathbb{R}$ is continuous (E having subspace topology from $\mathbb{R}$),
$f(D)$ need not be a Lebesgue null set for each $D \subset E$.

15) Let $\mathcal{N}(\mathbb{R}^d)$ be the collection of all Lebesgue null sets in $\mathbb{R}^d$. Then which of the following statements about $\mathcal{N}(\mathbb{R}^d)$ are true? *1 point*

- $\mathcal{N}(\mathbb{R}^d)$ is closed under taking lim sup and lim inf
- if $A \in \mathcal{N}(\mathbb{R}^d)$ then its power set is a subset of $\mathcal{N}(\mathbb{R}^d)$.
- There exists an injective map from $\mathcal{N}(\mathbb{R}^d)$ into $\mathbb{R}$

No, the answer is incorrect.
Score: 0

Accepted Answers:
$\mathcal{N}(\mathbb{R}^d)$ is closed under taking lim sup and lim inf
if $A \in \mathcal{N}(\mathbb{R}^d)$ then its power set is a subset of $\mathcal{N}(\mathbb{R}^d)$.