Unit 13 - Week 11

Assignment 11

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-12-02, 23:59 IST.

Notation: For a measurable function $f : \mathbb{R}^d \to \mathbb{C}$, we denote by Mf the associated Hardy-Littlewood maximal function

$$Mf(x) := sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f| dm$$

Definition:[Lebesgue point of density] Let $E \subset \mathbb{R}^d$ be Lebesgue measurable.We call a point $x \in \mathbb{R}^d$ a Lebesgue point of density for E if $\lim_{r\to 0^+} \frac{m(E \cap B(x,r))}{m(B(x,r))} = 1$.

Answer the following two questions based on this definition.

1) Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set of strictly positive measure.Then which of the folowing are true? [Hint:First assume that $m(E) < \infty$ and apply Lebesgue's differentiation theorem.] *1 point*

- Almost every point of E is a Lebesgue point of density for E
- It may hold that no point of E is a Lebesgue point of density for E
- Almost every point of E^c is not a Lebesgue point of density for E

No, the answer is incorrect.
Score: 0
Accepted Answers:
Almost every point of E is a Lebesgue point of density for E
Almost every point of E^c is not a Lebesgue point of density for E

2) Let $E \subset \mathbb{R}^d$ be Lebesgue measurable. If there exist no points of density for E, then E is a Lebesgue null set. *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

3) Let $f \in L^1(\mathbb{R}^d, m)$. Then the value of $\lim_{n\to\infty} m(\{x \in \mathbb{R}^d : Mf(x) \geq n^2\})$ is *1 point*

- 0
- 1
- ∞

No, the answer is incorrect.
Score: 0
Accepted Answers:
0

4) Let $\{f_n\}$ be a sequence of measurable functions $f_n : \mathbb{R}^d \to \mathbb{C}$ and let $f : \mathbb{R}^d \to \mathbb{C}$ be measurable such that $\sum_{n=1}^{\infty} ||f - f_n||_{L^1}$ converges. *1 point*

Then the following holds: $M(f - f_n) \to 0$ as $n \to \infty$ pointwise a.e.

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

5) Let $f \in L^1(\mathbb{R}^d)$. For $x \in \mathbb{R}$ define $G_x : [0, \infty) \to \mathbb{C}$ by *1 point*

$$G_x(r) := \begin{cases} \frac{1}{m(B(x,r))} \int_{B(x,r)} f\, dm & if\, r > 0 \\ f(x) & if\, r = 0 \end{cases}$$

Then

- G_x is continuous on $[0, \infty)$ for almost every $x \in \mathbb{R}^d$
- For almost every $x \in \mathbb{R}^d$, G_x is discontinuous at 0.
- If f is bounded, G_x is also bounded.

No, the answer is incorrect.
Score: 0
Accepted Answers:
G_x is continuous on $[0, \infty)$ for almost every $x \in \mathbb{R}^d$
If f is bounded, G_x is also bounded.

6) Let $f \in L^1(\mathbb{R}^d)$. Define the Lebesgue set L_f of f by $L_f := \{x \in \mathbb{R}^d : f(x) < \infty$ and $\lim_{r\to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy = 0\}$. Then *1 point*

- $\mu(L_f) = 0$
- Almost every $x \in \mathbb{R}^d$ is in L_f.
- L_f is Lebesgue measurable.

No, the answer is incorrect.
Score: 0
Accepted Answers:
Almost every $x \in \mathbb{R}^d$ is in L_f.
L_f is Lebesgue measurable.

7) Let $f \in L^1(\mathbb{R}^d)$ and let $x \in \mathbb{R}^d$ be a point of continuity of f . If L_f is defined as in previous question, then $x \in L_f$. *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True