Unit 12 - Week 10

Assignment 10

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-11-25, 23:59 IST.

Notation: If $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ are measure spaces, $\mathcal{B}_X \times \mathcal{B}_Y$ will denote the product σ−algebra and we shall call the Hahn-Kolmogrov extension of the elementary product measure as the product measure $\mu_X \times \mu_Y$ on $\mathcal{B}_X \times \mathcal{B}_Y$ (as done in lectures.)

1) Let $\mathcal{L}(\mathbb{R}^d)$ be the Lebesgue $\sigma-$ algebra on $\mathbb{R}^d$. Which of the following are true? *1 point*

- $\mathcal{L}(\mathbb{R}^d \times \mathbb{R}^{d'}) = \mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^{d'})$
- $\mathcal{L}(\mathbb{R}^d \times \mathbb{R}^{d'})$ is strictly included in $\mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^{d'})$.
- $\mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^{d'})$ is strictly included in $\mathcal{L}(\mathbb{R}^d \times \mathbb{R}^{d'})$.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^{d'})$ is strictly included in $\mathcal{L}(\mathbb{R}^d \times \mathbb{R}^{d'})$.

2) Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel $\sigma-$ algebra on $\mathbb{R}^d$. Which of the following are true? *1 point*

- $\mathcal{B}(\mathbb{R}^d \times R^{d'}) = \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d'})$
- $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d'})$ is strictly included in $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d'})$.
- $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d'})$ is strictly included in $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d'})$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathcal{B}(\mathbb{R}^d \times R^{d'}) = \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d'})$

3) Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be measure spaaces, and consider the product measure space $(X \times Y, \mathcal{B}_X \times \mathcal{B}_Y, \mu_X \times \mu_Y)$. Let $f : X \times Y \to [0, \infty]$ be an unsigned measurable function with respect to $\mathcal{B}_X \times \mathcal{B}_Y$. Then: *0 points*

- $f_x : Y \to [0, \infty]$ defined as $f_x(y) := f(x, y)$ is $\mathcal{B}_Y$- measurable for every $x \in X$.
- $f_x : X \to [0, \infty]$ defined as $f_y(x) := f(x, y)$ is $\mathcal{B}_Y$ - measurable for every $y \in Y$.
- If $\{g_x\}_{x \in X}$ is a collection of unsigned $\mathcal{B}_Y$ - measurable functions $g_x : Y \to [0, \infty]$, then $g : X \times Y \to [0, \infty]$ defined as $g(x, y) := g_x(y)$ is $\mathcal{B}_X \times \mathcal{B}_Y$ - measurable.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$f_x : Y \to [0, \infty]$ defined as $f_x(y) := f(x, y)$ is $\mathcal{B}_Y$- measurable for every $x \in X$.
$f_x : X \to [0, \infty]$ defined as $f_y(x) := f(x, y)$ is $\mathcal{B}_Y$ - measurable for every $y \in Y$.

4) Let $X = Y = [0, \infty)$ and $\mathcal{B}_X = \mathcal{B}_Y = \mathcal{B}([0, \infty))$ (The Borel σ- algebra). Let $f : X \times Y \to \mathbb{R}$ be defined as *1 point*

$$f(x, y) := \begin{cases} +1 & if\ \ 0 \le x < y,\ y > 0 \\ -1 & if\ \ y \le x \le 2y,\ y > 0 \\ 0 & otherwise \end{cases}$$

Let $\mu_X = \mu_Y = m$ be the Lebesgue measure on $[0, \infty)$. Then

- $f \in L^1(X \times Y, \mu_X \times \mu_Y)$
- $\int_X \int_Y f\, d\mu_Y d\mu_X$ and $\int_Y \int_X f d\mu_X d\mu_Y$ exist (i.e. are finite) and have equal value.
- $\int_X \int_Y f\, d\mu_Y d\mu_X$ exists but $\int_Y \int_X f\, d\mu_X d\mu_Y$ does not exist
- $\int_Y \int_X f\, d\mu_X d\mu_Y$ exists but $\int_X \int_Y f\, d\mu_Y d\mu_X$ does not exist

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\int_Y \int_X f\, d\mu_X d\mu_Y$ exists but $\int_X \int_Y f\, d\mu_Y d\mu_X$ does not exist

Let $(X, \mathcal{B}, \mu)$ be a σ- finite measure space and let $\mathbb{R}$ be equipped with the Borel σ- algebra and the (restricted) Lebesgue measure m. Let $f : X \to [0, \infty]$ be in $L^1(X, \mu)$. Let $E_f = \{(x, t) \in X \times \mathbb{R} : 0 \le t \le f(x)\}$. Answer the next two questions.

5) The set E_f is $\mathcal{B} \times \mathcal{B}(\mathbb{R})$ - measurable. *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

6) The value of $(\mu \times m)(E_f)$ is *1 point*

- $\int_X f\, d\mu$
- $\int_{X \times \mathbb{R}} \chi_{E_f} d(\mu \times m)$
- $\int_{[0,\infty)} \mu(\{x \in X : f(x) \ge \lambda\}) d\lambda$
- not well defined

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\int_X f\, d\mu$
$\int_{X \times \mathbb{R}} \chi_{E_f} d(\mu \times m)$
$\int_{[0,\infty)} \mu(\{x \in X : f(x) \ge \lambda\}) d\lambda$

Let $X = Y = [0, 1], \mathcal{B}_X = \mathcal{L}([0, 1]), \mathcal{B}_Y = \mathcal{P}([0, 1])$. Let $\mu_X = m$ (Lebesgue measure) and μ_Y = counting measure. Also let $D = \{(x, y) \in X \times Y : x = y\}$. Answer the following two questions.

7) The set D is $\mathcal{B}_X \times \mathcal{B}_Y$ - measurable. *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

8) The value of $(\mu_X \times \mu_Y)(D)$ is: *1 point*

- 1
- 0
- $+\infty$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$+\infty$

9) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{B}', \nu)$ be measure spaces. Let $\mu \times \nu$ be the product measure on $X \times Y$. Which of the following are true? *1 point*

- If $A \in \mathcal{B}$ and $B \in \mathcal{B}'$ then $\mu \times \nu(A \times B) = \mu(A)\nu(B)$.
- If μ and ν are σ −finite, $\mu \times \nu$ is σ −finite.
- If μ and ν are finite measures, $\mu \times \nu$ is a finite measure.
- $N \subset X \times Y$ is $\mu \times \nu$ measurable if and only if $N = A \times B$ for some $A \in \mathcal{B}$ and $B \in \mathcal{B}'$

No, the answer is incorrect.
Score: 0
Accepted Answers:
If $A \in \mathcal{B}$ and $B \in \mathcal{B}'$ then $\mu \times \nu(A \times B) = \mu(A)\nu(B)$.
If μ and ν are σ −finite, $\mu \times \nu$ is σ −finite.
If μ and ν are finite measures, $\mu \times \nu$ is a finite measure.

10) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{B}', \nu)$ be measure spaces. Let $\mu \times \nu$ be the product measure on $X \times Y$. Which of the following statements are true? *1 point*

- If $f : X \to \mathbb{C}$ is $\mathcal{B}$- measurable and $g : Y \to \mathbb{C}$ is $\mathcal{B}'$ measurable, then $f \times g : X \times Y \to \mathbb{C}$ defined by $f \times g(x, y) := f(x) + g(y)$ is measurable.
- If $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{B}, \mu)$ are complete measure spaces, then $(X \times Y, \mathcal{B} \times \mathcal{B}', \mu \times \nu)$ is a complete measure space.
- $(\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}), m \times m)$ is a complete measure space.

No, the answer is incorrect.
Score: 0
Accepted Answers:
If $f : X \to \mathbb{C}$ is $\mathcal{B}$- measurable and $g : Y \to \mathbb{C}$ is $\mathcal{B}'$ measurable, then $f \times g : X \times Y \to \mathbb{C}$ defined by $f \times g(x, y) := f(x) + g(y)$ is measurable.

11) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{B}', \nu)$ be measure spaces. Let $\mu \times \nu$ be the product measure on $X \times Y$. If $E \subset X \times Y$ is $\mathcal{B} \times \mathcal{B}'$ measurable, then $E_x = \{x \in X : (x, y) \in E\}$ is $\mathcal{B}$ measurable. *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
True

12) If X and Y are any two non empty sets, and $\mathcal{B}_X = \mathcal{P}(X)$ (power set) and $\mathcal{B}_Y = \mathcal{P}(Y)$, then the product σ algebra is always $\mathcal{P}(X \times Y)$ *1 point*

- True
- False

No, the answer is incorrect.
Score: 0
Accepted Answers:
False

13) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{B}', \nu)$ be measure spaces. Let $\mu \times \nu$ be the product measure on $X \times Y$. Which of the following statements are true? *1 point*

- $\mathcal{B} \times \mathcal{B}'$ is the smallest $\sigma-$ algebra $\mathcal{A}$ on $X \times Y$ such that $E \times F \in \mathcal{A}$ whenever $E \in \mathcal{B}$ and $F \in \mathcal{B}'$.
- $\mathcal{B} \times \mathcal{B}'$ is the smallest σ-algebra on $X \times Y$ that makes the projection maps $\pi_X : X \times Y \to X$ $(\mathcal{B} \times \mathcal{B}', \mathcal{B})$- measurable , and $\pi_Y : X \times Y \to Y$ $(\mathcal{B} \times \mathcal{B}', \mathcal{B}')$- measurable respectively.
- If X and Y are σ−finite, then the measure γ on $\mathcal{B} \times \mathcal{B}'$ satisfying $\gamma(E \times F) = \mu(E)\nu(F)$ for $E \in \mathcal{B}$ and $F \in \mathcal{B}'$ is unique.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathcal{B} \times \mathcal{B}'$ is the smallest $\sigma-$ algebra $\mathcal{A}$ on $X \times Y$ such that $E \times F \in \mathcal{A}$ whenever $E \in \mathcal{B}$ and $F \in \mathcal{B}'$.
$\mathcal{B} \times \mathcal{B}'$ is the smallest σ-algebra on $X \times Y$ that makes the projection maps $\pi_X : X \times Y \to X$ $(\mathcal{B} \times \mathcal{B}', \mathcal{B})$- measurable , and $\pi_Y : X \times Y \to Y$ $(\mathcal{B} \times \mathcal{B}', \mathcal{B}')$- measurable respectively.
If X and Y are σ−finite, then the measure γ on $\mathcal{B} \times \mathcal{B}'$ satisfying $\gamma(E \times F) = \mu(E)\nu(F)$ for $E \in \mathcal{B}$ and $F \in \mathcal{B}'$ is unique.