NPTEL » Measure Theory

Announcements About the Course Ask a Question Progress Mentor

## Unit 12 - Week 10

How does an NPTEL online

Course outline

course work?

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Week 2

Week 3

Week 4

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Week 9

Week 10

spaces

Theorems of Tonelli and Fubini-

discussion of product measure

Tonelli's theorem for sets - Part

Tonelli's theorem for sets - Part

integration for measurable and L^1 functions on sigma-finite

Fubini-Tonelli theorem:

interchanging order of

Quiz : Assignment 10

Week 10 Feedback Form

Assignment 10 solutions

measure spaces

Measure Theory

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Week 11

Week 12

interchanging the order of integration for repeated

integrals: motivation and

Product measures

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Assignment 10
   The due date for submitting this assignment has passed.
                                                                                                                                                                            Due on 2020-11-25, 23:59 IST.
   As per our records you have not submitted this assignment.
   Notation: If (X, \mathcal{B}_X, \mu_X) and (Y, \mathcal{B}_Y, \mu_Y) are measure spaces, \mathcal{B}_X \times \mathcal{B}_Y will denote the product \sigma-algebra and we shall call the Hahn-Kolmogrov extension
   of the elementary product measure as the product measure \mu_X \times \mu_Y on \mathcal{B}_X \times \mathcal{B}_Y (as done in lectures.)
   1) Let \mathcal{L}(\mathbb{R}^d) be the Lebesgue \sigma- algebra on \mathbb{R}^d . Which of the following are true?
                                                                                                                                                                                                                            1 point
      \mathcal{L}(\mathbb{R}^d 	imes \mathbb{R}^{d'}) = \mathcal{L}(\mathbb{R}^d) 	imes \mathcal{L}(\mathbb{R}^{d'})
      \mathcal{L}(\mathbb{R}^d 	imes \mathbb{R}^{d'}) is strictly included in \mathcal{L}(\mathbb{R}^d) 	imes \mathcal{L}(\mathbb{R}^{d'}).
      \mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^{d'}) is strictly included in \mathcal{L}(\mathbb{R}^d \times \mathbb{R}^{d'}).
   No, the answer is incorrect.
   Accepted Answers:
   \mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^{d'}) is strictly included in \mathcal{L}(\mathbb{R}^d \times \mathbb{R}^{d'}).
   2) Let \mathcal{B}(\mathbb{R}^d) be the Borel \sigma- algebra on \mathbb{R}^d . Which of the following are true?
                                                                                                                                                                                                                            1 point
      \mathcal{B}(\mathbb{R}^d 	imes R^{d'}) = \mathcal{B}(\mathbb{R}^d) 	imes \mathcal{B}(\mathbb{R}^{d'})
      \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d'}) is strictly included in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d'}).
      \mathcal{B}(\mathbb{R}^d) 	imes \mathcal{B}(\mathbb{R}^{d'}) is strictly included in \mathcal{B}(\mathbb{R}^d 	imes \mathbb{R}^{d'})
   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   \mathcal{B}(\mathbb{R}^d 	imes R^{d'}) = \mathcal{B}(\mathbb{R}^d) 	imes \mathcal{B}(\mathbb{R}^{d'})
  3) Let (X, \mathcal{B}_X, \mu_X) and (Y, \mathcal{B}_Y, \mu_Y) be measure spaaces, and consider the product measure space (X \times Y, \mathcal{B}_X \times \mathcal{B}_Y, \mu_X \times \mu_Y). Let
                                                                                                                                                                                                                          0 points
f: X 	imes Y 	o [0, \infty] be an unsigned measurable function with respect to \mathcal{B}_X 	imes \mathcal{B}_Y . Then:
     f_x:Y	o [0,\infty] defined as f_x(y):=f(x,y) is \mathcal{B}_{Y^-} measurable for every x\in X.
      f_x:X	o [0,\infty] defined as f_y(x):=f(x,y) is \mathcal{B}_Y - measurable for every y\in Y .
      \mathsf{If}\left\{g_x\right\}_{x\in X}\mathsf{is a collection of unsigned }\mathcal{B}_Y\mathsf{-measurable functions }g_x:Y\to[0,\infty],\mathsf{then }g:X\times Y\to[0,\infty]\mathsf{ defined as }g(x,y)\mathrel{\mathop:}=g_x(y)\mathsf{ is }\mathcal{B}_X\times\mathcal{B}_Y\mathsf{-measurable functions }g_x:Y\to[0,\infty]
   No, the answer is incorrect.
   f_x:Y	o [0,\infty] defined as f_x(y):=f(x,y) is \mathcal{B}_Y- measurable for every x\in X.
   f_x:X	o [0,\infty] defined as f_y(x):=f(x,y) is \mathcal{B}_Y - measurable for every y\in Y
  4) Let X=Y=[0,\infty) and \mathcal{B}_X=\mathcal{B}_Y=\mathcal{B}([0,\infty)) (The Borel \sigma- algebra). Let f:X	imes Y	o \mathbb{R} be defined as
                                                                                                                                                                                                                            1 point
                   f(x,y) \coloneqq egin{cases} +1 & if & 0 \leq x < y, \ y > 0 \ -1 & if & y \leq x \leq 2y, \ y > 0 \ 0 & otherwise \end{cases}
Let \mu_X = \mu_Y = m be the Lebesgue measure on [0,\infty) . Then
     f \in L^1(X 	imes Y, \mu_X 	imes \mu_Y)
     \int_X \int_Y \ f \ d\mu_Y d\mu_X and \int_Y \int_X f d\mu_X d\mu_Y exist (i.e. are finite) and have equal value.
     \int_X \int_Y \ f \ d\mu_Y d\mu_X exists but \int_Y \int_X \ f \ d\mu_X d\mu_Y does not exist
     \int_Y \int_X f \, d\mu_X d\mu_Y exists but \int_X \int_Y f \, d\mu_Y d\mu_X does not exist
   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   \int_{Y}\int_{X}^{\cdot}f\,d\mu_{X}d\mu_{Y} exists but \int_{X}\int_{Y}^{\cdot}f\,d\mu_{Y}d\mu_{X} does not exist
   Let (X, \mathcal{B}, \mu) be a \sigma- finite measure space and let \mathbb{R} be equipped with the Borel \sigma- algebra and the (restricted) Lebesgue measure m. Let f: X \to [0, \infty] be in
   L^1(X,\mu). Let E_f=ig\{(x,t)\in X	imes \mathbb{R}: 0\leq t\leq f(x)ig\}. Answer the next two questions.
  5) The set E_f is \mathcal{B} 	imes \mathcal{B}(\mathbb{R}) - measurable.
                                                                                                                                                                                                                            1 point
      True

    False

   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   6) The value of (\mu \times m)(E_f) is
                                                                                                                                                                                                                            1 point
      \int_X f d\mu
      \int_{X	imes\mathbb{R}}\chi_{E_f}d(\mu	imes m)
      \int_{[0,\infty)} \muig(ig\{x\in X: f(x)\geq \lambdaig\}ig)d\lambda
      not well defined
   No, the answer is incorrect.
   Score: 0
    Accepted Answers:
   \int_X f d\mu
   \int_{X	imes\mathbb{R}}\chi_{E_f}d(\mu	imes m)
   \int_{[0,\infty)} \muig(ig\{x\in X: f(x)\geq \lambdaig\}ig)d\lambda
  Let X=Y=[0,1], \mathcal{B}_X=\mathcal{L}([0,1]), \mathcal{B}_Y=\mathcal{P}([0,1]). Let \mu_X=m (Lebesgue measure) and \ \mu_Y = counting measure. Also let
   D = ig\{ (x,y) \in X 	imes Y : x = y ig\} . Answer the following two questions.
   7) The set D is \mathcal{B}_X \times \mathcal{B}_Y - measurable.
                                                                                                                                                                                                                            1 point
      True

    False

   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   8) The value of (\mu_X 	imes \mu_Y)(D) is:
                                                                                                                                                                                                                            1 point
      1
      \bigcirc 0
      +\infty
   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   +\infty
   9) Let (X, \mathcal{B}, \mu) and (Y, \mathcal{B}', \nu) be measure spaces. Let \mu \times \nu be the product measure on X \times Y. Which of the following are true?
                                                                                                                                                                                                                            1 point
     If A \in \mathcal{B} and B \in \mathcal{B}' then \mu 	imes 
u(A 	imes B) = \mu(A) 
u(B) .
      If \mu and \nu are \sigma -finite, \mu \times \nu is \sigma -finite.
      If \mu and \nu are finite measures, \mu \times \nu is a finite measure.
     N\subset X	imes Y is \mu	imes 
u measurable if and only if N=A	imes B for some A\in \mathcal{B} and B\in \mathcal{B}'
   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   If A\in\mathcal{B} and B\in\mathcal{B}' then \mu	imes 
u(A	imes B)=\mu(A)
u(B) .
   If \mu and \nu are \sigma -finite, \mu \times \nu is \sigma -finite.
   If \mu and \nu are finite measures, \mu \times \nu is a finite measure.
   10) Let (X, \mathcal{B}, \mu) and (Y, \mathcal{B}', \nu) be measure spaces. Let \mu \times \nu be the product measure on X \times Y. Which of the following statements are true?
                                                                                                                                                                                                                            1 point
     \text{If } f: X \to \mathbb{C} \text{ is } \mathcal{B}\text{- measurable and } g: Y \to \mathbb{C} \text{ is } \mathcal{B}' \text{ measurable, then } f \times g: X \times Y \to \mathbb{C} \text{ defined by } f \times g(x,y) := f(x) + g(y) \text{ is measurable.}
     If (X, \mathcal{B}, \mu) and (X, \mathcal{B}, \mu) are complete measure spaces, then (X \times Y, \mathcal{B} \times \mathcal{B}', \mu \times \nu) is a complete measure space.
     (\mathbb{R}^2,\mathcal{L}(\mathbb{R})	imes\mathcal{L}(\mathbb{R}),m	imes m) is a complete measure space.
   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   If f:X	o\mathbb{C} is \mathcal{B}- measurable and g:Y	o\mathbb{C} is \mathcal{B}' measurable, then f	imes g:X	imes Y	o\mathbb{C} defined by
   f 	imes g(x,y) := f(x) + g(y) is measurable.
  11) Let (X,\mathcal{B},\mu) and (Y,\mathcal{B}',\nu) be measure spaces. Let \mu \times \nu be the product measure on X \times Y. If E \subset X \times Y is \mathcal{B} \times \mathcal{B}' measurable, then
                                                                                                                                                                                                                            1 point
E_x = \{x \in X : (x,y) \in E\} is {\mathcal B} measurable.
      True

    False

   No, the answer is incorrect.
   Score: 0
   Accepted Answers:
   12) If X and Y are any two non empty sets, and \mathcal{B}_X = \mathcal{P}(X) (power set) and \mathcal{B}_Y = \mathcal{P}(Y), then the product \sigma algebra is always \mathcal{P}(X \times Y)
                                                                                                                                                                                                                            1 point
      True
      False
   No, the answer is incorrect.
   Accepted Answers:
   13) Let (X, \mathcal{B}, \mu) and (Y, \mathcal{B}', \nu) be measure spaces. Let \mu \times \nu be the product measure on X \times Y. Which of the following statements are true?
                                                                                                                                                                                                                            1 point
      \mathcal{B}	imes\mathcal{B}' is the smallest \sigma- algebra \mathcal{A} on X	imes Y such that E	imes F\in \mathcal{A} whenever E\in \mathcal{B} and F\in \mathcal{B}'
      \mathcal{B} 	imes \mathcal{B}' is the smallest \sigma-algebra on X 	imes Y that makes the projection maps \pi_X : X 	imes Y 	o X (\mathcal{B} 	imes \mathcal{B}', \mathcal{B})- measurable , and \pi_Y : X 	imes Y 	o Y
      (\mathcal{B} \times \mathcal{B}', \mathcal{B}')- measurable respectively.
     If X and Y are \sigma-finite, then the measure \gamma on \mathcal{B} \times \mathcal{B}' satisfying \gamma(E \times F) = \mu(E)\nu(F) for E \in \mathcal{B} and F \in \mathcal{B}' is unique.
   No, the answer is incorrect.
   Score: 0
   \mathcal{B}	imes\mathcal{B}' is the smallest \sigma– algebra \mathcal A on X	imes Y such that E	imes F\in \mathcal A whenever E\in \mathcal B and F\in \mathcal B'
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 $\mathcal{B} \times \mathcal{B}'$  is the smallest  $\sigma$ -algebra on  $X \times Y$  that makes the projection maps  $\pi_X : X \times Y \to X$   $(\mathcal{B} \times \mathcal{B}', \mathcal{B})$ - measurable, and  $\pi_Y : X \times Y \to Y$   $(\mathcal{B} \times \mathcal{B}', \mathcal{B}')$ - measurable respectively.

and  $F \in \mathcal{B}'$  is unique.

If X and Y are  $\sigma$ -finite, then the measure  $\gamma$  on  $\mathcal{B} imes \mathcal{B}'$  satisfying  $\gamma(E imes F) = \mu(E) \nu(F)$  for  $E \in \mathcal{B}$