Assignment - 9

(1) Find the Solution of the initial Value problem

$$u_{tt} - 3u_{xx} = 0, \quad x \in \mathbb{R}, t > 0$$
$$u(x,0) = x \sin x, \quad x \in \mathbb{R}$$
$$u_t(x,0) = \cos 2x, \quad x \in \mathbb{R}.$$

Solution:

By D' Alembert's formula,

$$u(x,t) = \frac{1}{2}\left[(x-\sqrt{3}t)\sin(x-\sqrt{3}t) + (x+\sqrt{3}t)\sin(x+\sqrt{3}t)\right] + \frac{1}{2\sqrt{3}}\int_{x-\sqrt{3}t}^{x+\sqrt{3}t} \cos 2u \, du$$

$$= \frac{1}{2}\left[(x-\sqrt{3}t)\sin(x-\sqrt{3}t) + (x+\sqrt{3}t)\sin(x+\sqrt{3}t)\right] + \frac{1}{4\sqrt{3}}\left[\sin(x+\sqrt{3}t) - \sin(x-\sqrt{3}t)\right].$$

---

(2) Find the Solution of the initial boundary value problem

$$u_{tt} - 4u_{xx} = 0, \quad x > 0, t > 0$$

$$\text{Initial} \begin{cases} u(x,0) = \sin^2 x, & x \geq 0 \\ u_t(x,0) = 0, & x \geq 0. \end{cases}$$

$$\text{BC} \quad \{ u(0,t) = 0, \quad t \geq 0.$$

Solution: By using D' Alembert's formula, we obtain,

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\sin^2(x+2t) - \sin^2(x-2t)\right], & 0 \leq x < 2t \\ \frac{1}{2}\left[\sin^2(x+2t) + \sin^2(x-2t)\right], & x > 2t. \end{cases}$$

(3) Find the solution of the initial boundary value problem,

$$u_{tt} - 4u_{xx} = 0 \quad x>0, t>0$$

$$u(x,0) = \sin x, \quad x \geq 0$$

$$u_t(x,0) = x^2, \quad x \geq 0.$$

$$u(0,t) = 0, \quad t \geq 0.$$

Solution:

Again by the use of D'Alembert's formulae,

$$u(x,t) = \begin{cases} \frac{1}{2}\left[-\sin(2t-x) + \sin(2t+x)\right] + \frac{1}{4}\left(\frac{(x+2t)^3}{3} - \frac{(2t-x)^3}{3}\right), & 0<x<2t \\ \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t)\right] + \frac{1}{4}\left(\frac{(x+2t)^3}{3} - \frac{(x-2t)^3}{3}\right), & x>2t. \end{cases}$$

---

(4) Find the solution of the initial boundary value problem.

$$u_{tt} - u_{xx} = 0, \quad x>0, t>0$$

$$u(x,0) = \cos\left(\frac{\pi x}{2}\right), \quad x \geq 0$$

$$u_t(x,0) = 0, \quad x \geq 0$$

$$u_x(0,t) = 0, \quad t \geq 0.$$

Solution:

(using the formula obtained in the lecture).

the solution is

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\cos\left(\frac{\pi}{2}(t-x)\right) + \cos\left(\frac{\pi}{2}(t+x)\right)\right], & x<t \\ \frac{1}{2}\left[\cos\frac{\pi}{2}(x-t) + \cos\frac{\pi}{2}(x+t)\right], & x>t \end{cases}$$

$$u(x,t) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi}{2}t\right), \quad t>0, x>0.$$

---

⑤ Consider the pde $u_{tt} + \alpha u_{tx} + \beta u_{xx} = 0$, $x \in \mathbb{R}, t > 0$, where $\alpha, \beta \in \mathbb{R}$ such that $\lambda^2 + \alpha\lambda + \beta = 0$, has two distinct roots $c_1, c_2 \in \mathbb{R}$. Then, use the transformation variables $\xi = x + c_1 t$ and $\eta = x + c_2 t$ to reduce the above pde to $u_{\xi\eta} = 0$.

Solution: first note that $\alpha = -(c_1 + c_2)$; $\beta = c_1 c_2$.

$$u_t = u_\xi \xi_t + u_\eta \eta_t$$
$$= c_1 u_\xi + c_2 u_\eta$$

$$u_{tt} = c_1^2 u_{\xi\xi} + 2c_1 c_2 u_{\xi\eta} + c_2^2 u_{\eta\eta}$$

$$u_x = u_\xi + u_\eta$$

$$u_{xt} = c_1(u_{\xi\xi} + u_{\xi\eta}) + c_2(u_{\xi\eta} + u_{\eta\eta})$$
$$= c_1 u_{\xi\xi} + (c_1 + c_2) u_{\xi\eta} + c_2 u_{\eta\eta}.$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

now,

$$u_{tt} + \alpha u_{tx} + \beta u_{xx} = c_1^2 u_{\xi\xi} + 2c_1 c_2 u_{\xi\eta} + c_2^2 u_{\eta\eta} - (c_1 + c_2)\begin{bmatrix} c_1 u_{\xi\xi} + c_2 u_{\eta\eta} \\ + (c_1 + c_2) u_{\xi\eta} \end{bmatrix}$$
$$+ c_1 c_2 (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$= 2c_1 c_2 u_{\xi\eta} - c_1 c_2 u_{\xi\xi} - c_1 c_2 u_{\eta\eta} - (c_1^2 + c_2^2 + 2c_1 c_2) u_{\xi\eta}$$
$$+ c_1 c_2 u_{\xi\xi} + c_1 c_2 u_{\eta\eta} + 2c_1 c_2 u_{\xi\eta}$$

$$= -(c_1 - c_2)^2 u_{\xi\eta}$$

Therefore $(c_1 - c_2)^2 u_{\xi\eta} = 0$, since $c_1 \neq c_2$ (distinct)

implies $u_{\xi\eta} = 0$

(6) Find the Solution of the pde $u_{tt} + \alpha u_{tx} + \beta u_{xx} = 0$, $x \in \mathbb{R}, t > 0$, where $\alpha, \beta \in \mathbb{R}$ are such that $\lambda^2 + \alpha\lambda + \beta = 0$ has two distinct roots $c_1, c_2 \in \mathbb{R}$, with initial Conditions $u(x,0) = f(x)$

$u_t(x,0) = g(x)$, for $x \in \mathbb{R}$

Solution: As we have shown in problem (5), for given Such α, β, the pde can be equivalently written as

$$u_{\xi\eta}(\xi,\eta) = 0 \text{ , where } \xi = x + c_1 t \text{ , } \eta = x + c_2 t.$$

$$\Rightarrow u(\xi,\eta) = \phi_1(\xi) + \phi_2(\eta) \text{ ,}$$

interms of (x,t),

$$u(x,t) = \phi_1(x + c_1 t) + \phi_2(x + c_2 t)$$

Since $u(x,0) = f(x)$, $x \in \mathbb{R}$,

$$\phi_1(x) + \phi_2(x) = f(x) \quad \longrightarrow (1)$$

and $u_t(x,0) = g(x)$ $\forall x$

implies, $$\left.\frac{\partial \phi_1(x+c_1t)}{\partial(x+c_1t)} \cdot \frac{\partial(x+c_1t)}{\partial t}\right|_{t=0} + \left.\frac{\partial \phi_2(x+c_2t)}{\partial(x+c_2t)} \cdot \frac{\partial(x+c_2t)}{\partial t}\right|_{t=0} = g(x)$$

$$\Rightarrow c_1 \phi_{1x}(x) + c_2 \phi_{2x}(x) = g(x), \quad x \in \mathbb{R}$$

$$\Rightarrow \int_{x_0}^{x} g(s)\,ds = c_1 \int_{x_0}^{x} \phi_{1s}(s)\,ds + c_2 \int_{x_0}^{x} \phi_{2s}(s)\,ds$$

$$= c_1 \phi_1(x) + c_2 \phi_2(x) - c_1 \phi_1(x_0) - c_2 \phi_2(x_0)$$

$\longrightarrow (2)$

(6) Contd...

from (1) & (2), we have the following,

$$\phi_1(x) + \phi_2(x) = f(x)$$

& $$c_1\phi_1(x) + c_2\phi_2(x) = \int_{x_0}^{x} g(s)\,ds + c_1\phi_1(x_0) + c_2\phi_2(x_0).$$

$$\Rightarrow \quad \phi_2(x) = \frac{1}{(c_2-c_1)}\left[-c_1 f(x) + \int_{x_0}^{x} g(s)\,ds + c_1\phi_1(x_0) + c_2\phi_2(x_0)\right]$$

III^{ly} $$\phi_1(x) = \frac{1}{(c_1-c_2)}\left[-c_2 f(x) + \int_{x_0}^{x} g(s)\,ds + c_1\phi_1(x_0) + c_2\phi_2(x_0)\right]$$

Therefore,

$$u(x,t) = \phi_1(x+c_1t) + \phi_2(x+c_2t)$$

i.e., $$u(x,t) = \frac{c_2}{c_2-c_1} f(x+c_1t) + \frac{c_1}{c_1-c_2} f(x+c_2t)$$

$$+ \frac{1}{c_1-c_2}\int_{x_0}^{x+c_1t} g(s)\,ds + \frac{1}{c_2-c_1}\int_{x_0}^{x+c_2t} g(s)\,ds.$$

$t > 0,\ x \in \mathbb{R}$

is the general solution