NPTEL COURSE - Introduction to Commutative Algebra

Assignment solution - Week 9

(1) Let A be a subring of a ring B such that the set $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Solution. Suppose A is not integrally closed in B. Then there is an element b in $B \setminus A$ such that b is integral over A. Let $f(x) = a_0 + a_1 x + \cdots + x^n \in A[x]$ be of smallest degree monic polynomial such that $f(b) = 0$. We have

$$b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0.$$

Note that $n \geq 2$ and $b \notin A$. By minimality of n, $b^{n-1} + a_{n-1} b^{n-2} + \cdots + a_1 \notin A$. But we have

$$b(b^{n-1} + a_{n-1} b^{n-2} + \cdots + a_1) = -a_0 \in A.$$

This a contradiction to $B \setminus A$ is closed under multiplication.

(2) Let A be a commutative ring and M be a Noetherian A-module. Prove that $A/\operatorname{Ann}(M)$ is a Noetherian ring.

Solution. Since M is Noetherian A-module, M is a finitely generated over A. Let $\{m_1, \ldots, m_n\}$ be generating set of M. Consider the ring homomorphism

$$\phi : A \to \oplus_{i=1}^{n} M$$

and defined by $\phi(1) = (m_1, \ldots, m_n)$. We can easy to check $\ker \phi = \operatorname{Ann}(M)$. Therefore

$$A/\ker \phi = A/\operatorname{Ann}(M) \cong \phi(A).$$

Since $\oplus_{i=1}^{n} M$ is a Noetherian, $\phi(A)$ is a Noetherian. Hence $A/\operatorname{Ann}(M)$ is a Noetherian A-module. If $\bar{I}$ is an ideal of $A/\operatorname{Ann}(M)$, then $\bar{I}$ is an A-submodule of $A/\operatorname{Ann}(M)$. Therefore $\bar{I}$ is finitely generated as an A-submodule and hence finitely generated as an ideal. Therefore $A/\operatorname{Ann}(M)$ is a Noetherian ring.

(3) If A is Noetherian, then prove that any surjective homomorphism $\phi : A \to A$ is an isomorphism.

Solution. Consider the increasing chain of ideals of A:

$$\ker \phi \subseteq \ker \phi^2 \subseteq \cdots \subseteq \ker \phi^n \subseteq \cdots .$$

Since A is Noetherian, there exists $n_0 \in \mathbb{N}$ such that $\ker \phi^n = \ker \phi^{n+1}$ for all $n \geq n_0$, i.e., $\phi^n(a) = 0$ if and only if $\phi(\phi^n(a)) = 0$ for all $n \geq n_0$. Let $b \in \ker \phi$. Let $b_1 \in A$ be such that $\phi(b_1) = b$. Let b_2 be such that $\phi(b_2) = b_1$ and hence $\phi^2(b_2) = b$. For each $n \geq 1$, let $b_n \in A$ be such that $\phi^n(b_n) = b$. Let $n \geq n_0$. Then $\phi^{n+1}(b_n) = \phi(\phi^n(b_n)) = 0$ and hence $\phi^n(b_n) = b = 0$. Therefore $\ker \phi = 0$. Hence ϕ is injective.

(4) Find a composition series for $\mathbb{Z}/30\mathbb{Z}$.

Solution. We start by looking for a maximal subgroup say $m\mathbb{Z}/n\mathbb{Z}$. Note that m is a prime divisor of n and $\mathbb{Z}_n/m\mathbb{Z}_n \cong \mathbb{Z}_m$. Now repeat the process. In this case, we have

$$\mathbb{Z}/30\mathbb{Z} \supset 2\mathbb{Z}/30\mathbb{Z} \supset 6\mathbb{Z}/30\mathbb{Z} \supset \bar{0}.$$