NPTEL COURSE - Introduction to Commutative Algebra

Assignment solution- Week 8

(1) Let A be a subring of B such that B is integral over A. Let $\mathfrak{n}$ be a maximal ideal of A. Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$, where $\mathfrak{m} = \mathfrak{n} \cap A$?

Solution : Let $A = k[x^2 - 1]$ and $B = k[x]$. Then B is integral over A. Note that $\mathfrak{n} = (x - 1)$ is a maximal ideal in B and $\mathfrak{m} = (x^2 - 1)$ is a maximal ideal in A with $\mathfrak{m} = \mathfrak{n} \cap A$. Suppose $\frac{1}{x+1} \in B_{\mathfrak{n}}$ is integral over $A_{\mathfrak{m}}$. Then there exists $n > 0$, $\frac{a_i}{s_i} \in A_{\mathfrak{m}}$ such that

$$\frac{1}{(x+1)^n} + \frac{a_{n-1}}{s_{n-1}} \frac{1}{(x+1)^{n-1}} + \cdots + \frac{a_0}{s_0} = 0.$$

After clearing the denominators we get

$$z \sum_{i=0}^{n} a_i t_i (x+1)^{n-i} = 0,$$

where $a_n = 1$, $t_i = s_{n-1} \cdots s_{i-1} s_{i+1} \cdots s_0$, $0 \leq i \leq n-1$, $t_n = s_{n-1} \cdots s_0$ and $z \in B \backslash \mathfrak{n}$. We have

$$z a_0 t_0 (x+1)^n + \cdots + z t_n = 0$$

which implies that

$$(x+1)\{z a_0 t_0 (x+1)^{n-1} + \cdots + z a_{n-1} t_{n-1}\} = -z t_n.$$

Therefore $x+1 | z t_n$. Since $x+1$ is a prime element, $x+1 | z$, $z \in (x+1)B \cap A = (x^2 - 1)$. This is contraction to $f \in B \setminus \mathfrak{n}$. Therefore $\frac{1}{(x+1)}$ is not integral over $A_{\mathfrak{m}}$.

(2) Let B be an integral extension of A. Prove that

(a) If $x \in A$ is a unit in B, then it is a unit in A.

Solution. Since B is integral over A, $1/x$ satisfies

$$\frac{1}{x^n} + a_1 \frac{1}{x^{n-1}} + \cdots + a_n = 0,$$

where $a_i \in A$, for $1 \leq i \leq n$. Now multiply by x^{n-1}, we get

$$\frac{1}{x} + a_1 + \cdots + a_{n-1} x^{n-2} + a_n x(n-1) = 0,$$

and

$$\frac{1}{x} = -(a_1 + \cdots + a_{n-1} x^{n-2} + a_n x(n-1) \in A.$$

Therefore x is a unit in A.

(b) the Jacobson radical of A is the contraction of the Jacobson radical of B.

Solution. Let J_A and J_B be Jacobson radical of A and B respectively. We need to prove that $J_A = J_B \cap A$. Let $x \in J_B \cap A$. Then for any $a \in A$, $1 + ax \in A$ is a unit in B. By (a), $1 + ax$ is a unit in A. Therefore $x \in J_A$.

Let $x \in J_A$. For a maximal ideal $\mathfrak{m}_B$ of B, $\mathfrak{n} = \mathfrak{m}_B \cap A$ is a maximal ideal in A. Therefore $x \in \mathfrak{n} = \mathfrak{m}_B \cap A \subseteq \mathfrak{m}_B$. Hence $x \in J_B \cap A$.

(3) If A is an integral domain, then prove that A is integrally closed in $A[x]$. Give an example of A such that A is not integrally closed in $A[x]$.

Solution. Suppose $f(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0 \in A[x] \setminus A$, $b_m \neq 0$, is integral over A. Then there is a monoic polynomial $g(t) = a_0 + a_1 t + \cdots a_{n-1}t^{n-1} + t^n \in A[t]$ such that $g(f(x)) = 0$. Therefore

$$b_m^n x^{mn} + \cdots + (a_n + a_{n-1}b_0^{n-1} + \cdots + b_0^n) = 0.$$

Hence $b_m^n = 0$ so that $b_m = 0$. This is a contradiction to the assumption that $b_m \neq 0$. Therefore A is integrally closed in $A[x]$.

Consider $A = \mathbb{Z}_4$ and $f(x) = 2x + 2 \in \mathbb{Z}_4[x]$. Note that $f(x)^2 = 0$. Therefore $f(x)$ is integral over A but $f(x) \notin A$.