NPTEL COURSE - Introduction to Commutative Algebra

Assignment solution - Week 7

(1) Suppose that for each prime ideal $\mathfrak{p} \subset A$, the local ring $A_{\mathfrak{p}}$ has no nonzero nilpotent elements. Prove that A has no nonzero nilpotent elements.

Solution: Suppose x is a nonzero nilpotent element. Note that, if x is nilpotent in A, then $x/1$ is also nilpotent in $A_{\mathfrak{q}}$. Therefore $\frac{x}{1} = 0 \in A_{\mathfrak{q}}$ for all prime ideals $\mathfrak{q}$. Since x is nonzero nilpotent, $(0 : x)$ is a proper ideal. Let $\mathfrak{p}$ be a prime containing $(0 : x)$. Since $\frac{x}{1} = 0 \in A_{\mathfrak{p}}$, there exists $s \in A \setminus \mathfrak{p}$ such that $sx = 0$, i.e., $s \in (0 : x) \subset \mathfrak{p}$ which is a contradiction. Hence $x = 0$.

(2) Let I be an ideal and let $S = 1 + I = \{1 + x \;:\; x \in I\}$. Prove that S is a multiplicatively closed subset. Prove that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}A$.

Solution: Note that $0 \notin S$ and $1 \in S$. If $1 + x, 1 + y \in S$, then $(1 + x)(1 + y) = 1 + (x + y + xy) \in S$. Therefore S is a multiplicatively closed subset.

Let $x/s \in S^{-1}I$, where $s \in S$ and $x \in I$. We need to show that x/s is in the Jacobson radical of $S^{-1}A$. For $r/t \in S^{-1}A$, where $t \in S$ and $r \in A$,

$$1 + \frac{r}{t}\frac{x}{s} = \frac{ts + rx}{ts}.$$

Since t and s are in $S = 1 + I$ and $rx \in I$, we see that $ts \in S$ and $ts + rx \in S$. Therefore we conclude that $(ts + rx)/(ts)$ is a unit in $S^{-1}A$. Hence $\frac{x}{s}$ is in the Jacobson radical of $S^{-1}A$.

(3) For two ideals I, J in A, prove that $I \subset J$ if and only if $I_{\mathfrak{m}} \subset J_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$ for all maximal ideal $\mathfrak{m}$.

Solution: If $I \subseteq J$, then $S^{-1}I \subset S^{-1}J$ in $S^{-1}A$ for any multiplicative set S.

Conversely, let $I_{\mathfrak{m}} \subset J_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$. Suppose $I \not\subseteq J$. Then $\frac{I+J}{J} \neq 0$ i.e., there exists an element $0 \neq z \in \frac{I+J}{J}$. Therefore, $(0 : z)$ is a proper ideal in A and hence contained in a maximal ideal, say $\mathfrak{m}$. We have

$$\left(\frac{I + J}{J}\right)_{\mathfrak{m}} \cong \frac{I_{\mathfrak{m}} + J_{\mathfrak{m}}}{J_{\mathfrak{m}}} = 0.$$

Therefore $z/1 = 0$. Then there exist a $t \in A \setminus \mathfrak{m}$ such that $tz = 0$. Hence $t \in (0 : z)$. This contradicts the assumption that $(0 : z) \cap (A \setminus \mathfrak{m}) = \emptyset$. Therefore $I \subseteq J$.

(4) Is $\sqrt{2 + \sqrt{2}} + \frac{1}{2}\sqrt[3]{3} \in \mathbb{R}$ integral over $\mathbb{Z}$? Justify your answer.

Solution : Let $C = \{x \in \mathbb{R} \mid x \text{ is integral over } \mathbb{Z}\}$. Then $-\sqrt{2 + \sqrt{2}} \in C$. Therefore, if $\sqrt{2 + \sqrt{2}} + \frac{1}{2}\sqrt[3]{3} \in C$, then $\frac{1}{2}\sqrt[3]{3} \in C$ and hence $\frac{3}{8} \in C$ which is a contradiction since $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$. Therefore $\sqrt{2 + \sqrt{2}} + \frac{1}{2}\sqrt[3]{3}$ is not integral over $\mathbb{Z}$.