NPTEL COURSE - Introduction to Commutative Algebra

Week 6 -Assignment solution

(1) Prove that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m, n)$.

Solution: *Recall*: $A/I \otimes_A M \cong M/IM$. Therefore we have

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}/n(\mathbb{Z}/m\mathbb{Z})$$

We need to show that $n(\mathbb{Z}/m\mathbb{Z}) = d(\mathbb{Z}/m\mathbb{Z}) \cong d\mathbb{Z}/m\mathbb{Z}$. Since d divides n, $n(\mathbb{Z}/m\mathbb{Z}) \subseteq d(\mathbb{Z}/m\mathbb{Z})$. Let $dx + m\mathbb{Z} \in d(\mathbb{Z}/m\mathbb{Z})$, where $x \in \mathbb{Z}$. Since $d = \gcd(m, n)$, $d = am + bn$, for some $a, b \in \mathbb{Z}$.

$$\begin{aligned} dx + m\mathbb{Z} &= (am + bn)x + m\mathbb{Z} \\ &= bnx + m\mathbb{Z} \in n(\mathbb{Z}/m\mathbb{Z}) \end{aligned}$$

Therefore $d(\mathbb{Z}/m\mathbb{Z}) \subseteq n(\mathbb{Z}/m\mathbb{Z})$. Hence $n(\mathbb{Z}/m\mathbb{Z}) = d(\mathbb{Z}/m\mathbb{Z})$.

(2) Let A be a local ring, M and N are finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then either $M = 0$ or $N = 0$.

Solution: First note that if V_1 and V_2 are non-zero vector spaces over a field k, then $V_1 \otimes_k V_2 \neq 0$, since if v_1 and v_2 are bases elements of V_1 and V_2 respectively, then $v_1 \otimes v_2$ is part of a basis for $V_1 \otimes_k V_2$.
Let A be a local ring with the maximal ideal $\mathfrak{m}$. We have

$$\begin{aligned} (M \otimes_A N) \otimes A/\mathfrak{m} &\cong (M \otimes_A N) \otimes_A (A/\mathfrak{m} \otimes_{A/\mathfrak{m}} A/\mathfrak{m}) \\ &\cong (M \otimes_A A/\mathfrak{m}) \otimes_{A/\mathfrak{m}} (N \otimes_A A/\mathfrak{m}) \end{aligned}$$

Since $M \otimes_A N = 0$, $(M \otimes_A A/\mathfrak{m}) \otimes_{A/\mathfrak{m}} (N \otimes_A A/\mathfrak{m}) = 0$. We know that $M \otimes_A A/\mathfrak{m} \cong M/\mathfrak{m}M$ and $N \otimes_A A/\mathfrak{m} \cong N/\mathfrak{m}N$. Therefore,

$$M/\mathfrak{m}M \otimes_{A/\mathfrak{m}} N/\mathfrak{m}N = 0.$$

Since $A/\mathfrak{m}$ is a field, $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are $A/\mathfrak{m}$-vector spaces. Therefore, either $M/\mathfrak{m}M = 0$ or $N/\mathfrak{m}N = 0$. If $M/\mathfrak{m}M = 0$, then by Nakayama's Lemma, $M = 0$. Similarly, if $N/\mathfrak{m}N = 0$, then $N = 0$.

(3) A multiplicatively closed subset S of A is said to be *saturated* if $xy \in S \Leftrightarrow x \in S$ and $y \in S$. Prove that S is saturated if and only if $A \setminus S$ is a union of prime ideals.

Solution: *Recall:* For any ideal I in A with $I \cap S = \emptyset$, then there is a prime ideal $\mathfrak{p}$ containing I such that $\mathfrak{p} \cap S = \emptyset$.

Suppose S is saturated. If $x \in A \setminus S$, then $S \cap (x) = \emptyset$. Therefore, there exists a prime ideal $\mathfrak{p}_x$ such that $(x) \subseteq \mathfrak{p}_x$ and $S \cap \mathfrak{p}_x = \emptyset$. Therefore $A \setminus S = \cup_{x \notin S} \mathfrak{p}_x$, i.e., $A \setminus S$ is a union of prime ideal of A.

Now suppose $A \setminus S = \cup_{\mathfrak{p}} \mathfrak{p}$. Then $S = \cap_{\mathfrak{p}} (A \setminus \mathfrak{p})$. For, $x, y \in A$

$$xy \in S \Leftrightarrow xy \notin \mathfrak{p} \text{ for all such primes } \Leftrightarrow x \notin \mathfrak{p} \ \& \ y \notin \mathfrak{p} \Leftrightarrow x \in S \ \& \ y \in S.$$

Therefore S is saturated set.

(4) Prove that if S is a multiplicatively closed subset of A, then $S^{-1}A$ is a flat A-module.

Solution: *Recall*: If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of A-modules, then $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact sequence of $S^{-1}A$-modules.

Let $0 \to M \to N$ be an exact sequence. Tensoring the exact sequence with $S^{-1}A$ we get $0 \to S^{-1}A \otimes_A M \to S^{-1}A \otimes_A N$. Also, $S^{-1}A \otimes_A M \cong S^{-1}M$ and $S^{-1}A \otimes_A N \cong S^{-1}N$. Therefore the tensored sequence becomes $0 \to S^{-1}M \to S^{-1}N$. Since this is an exact sequence, $S^{-1}A$ is a flat A-module.