NPTEL COURSE - Introduction to Commutative Algebra

Assignment - 5

(1) Let $M_1, M_2 \subset M$ be A-submodules of a given module M. Prove that if $M_1 + M_2$ and $M_1 \cap M_2$ are finitely generated, then so are M_1 and M_2.

Solution :

Lemma: Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be short exact sequence. If M' and M'' are finitely generated over A, then M is finitely generated over A.

Let N be a submodule of M. Then we have a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

By Lemma, if M/N and N are finitely generated over A, then M is finitely generated over A.

We have an isomorphism

$$\frac{M_1 + M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$$

Since $\frac{M_1+M_2}{M_1}$ is finitely generated over A, $\frac{M_2}{M_1 \cap M_2}$ is finitely generated over A. By Lemma M_2 is finitely generated over A. Similarly we can show that M_1 is finitely generated over A.

(2) Let A be a UFD and $x, y \in A$ be such that x and y does not have a common factor. Let $I = (x, y) \subset A$. Prove that the sequence $0 \longrightarrow A \xrightarrow{\phi} A^2 \xrightarrow{\psi} I \longrightarrow 0$ is exact, where $\phi(a) = (-ya, xa)$ and $\psi((a, b)) = ax + by$.

Solution : We need to show that ϕ is injective, ψ is surjective and $\text{im}(\phi) = \ker(\psi)$. It is not hard to verify that ϕ is injective and ψ is surjective. Let $(a, b) \in \text{im}(\phi)$. Then there exist an element $u \in A$ such that $\phi(u) = (a, b)$. Therefore $\phi(u) = (-yu, xu) = (a, b)$

$$\psi(a, b) = ax + by = (-yux + xuy) = 0.$$

Therefore $\text{im}(\phi) \subseteq \ker(\psi)$.

Let $(a, b) \in \ker(\psi)$. i.e., $\psi(a, b) = (ax + by) = 0$. Then $ax = -by$. Since A is a UFD and x, y have no common factor, $a = -cy$ and $b = dx$ for some $c, d \in A$. Then $(-c + d)xy = 0$, so $c = d$ and $(a, b) \in (-y, x)A$. Therefore $\ker(\psi) \subseteq \text{im}(\phi)$.

(3) Let $0 \to V_1 \to \cdots \to V_n \to 0$ be an exact sequence of finite dimensional vector spaces over a field k. Prove that $\sum_{i=1}^{n} (-1)^i \dim_k(V_i) = 0$.

Solution : Let

$$0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \to \cdots \to V_n \xrightarrow{f_n} 0$$

be an exact sequence. By the Rank-Nullity theorem, we have for all $1 \leq i \leq n$

$$\dim_k(V_i) = \dim_k(\ker(f_i)) + \dim_k(\operatorname{im}(f_i)).$$

We have,

$$\begin{aligned}
\dim_k(V_1) &= \dim_k(\operatorname{im}(f_1)) + \dim_k(\ker(f_1)) \\
\dim_k(V_1) &= \dim_k(\operatorname{im}(f_1)) \quad \text{(since } f_1 \text{ is an injective)} \\
\dim_k(V_2) &= \dim_k(\operatorname{im}(f_2)) + \dim_k(\ker(f_2)) \\
\dim_k(V_2) - \dim_k(V_1) &= \dim_k(\operatorname{im}(f_2)) \quad \text{(since } \ker(f_2) = \operatorname{im}(f_1) \text{)} \\
&\cdots \\
&\cdots \\
\dim_k(V_{n-1}) - \dim_k(V_{n-2}) + \cdots + (-1)^n \dim_k(V_1) &= \dim_k(\operatorname{im}(f_{n-1})) \\
&= \dim_k(V_n) \quad \text{(since } f_{n-1} \text{ is onto).}
\end{aligned}$$

Hence $\sum_{i=1}^{n}(-1)^i \dim_k(V_i) = 0$.

(4) Prove that $M \otimes N \cong N \otimes M$.

Solution : The map $M \times N \xrightarrow{\phi_1} N \otimes M$ given by

$$(x, y) \to y \otimes x$$

is a well defined A-bilinear map on $M \times N$, which can be extended to an A-linear map $M \otimes N \to N \otimes M$ given by $x \otimes y \to y \otimes x$. The map $N \times M \xrightarrow{\phi_2} M \otimes N$ given by

$$(y, x) \to x \otimes y$$

is a well defined A-bilinear map on $N \times M$, which can be extended to an A-linear map $N \otimes M \to M \otimes N$ given by $y \otimes x \to x \otimes y$.

We can prove that $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ are identity maps. Therefore $M \otimes N \cong N \otimes M$.