NPTEL COURSE - Introduction to Commutative Algebra

Assignment - 3

(1) Let I be an ideal in a commutative ring with identity, A, and $P_1, \ldots, P_r$ be prime ideals of A. Prove that if $I \subseteq \bigcup_{i=1}^{r} P_i$, then $I \subseteq P_i$ for some $1 \leq i \leq r$.

Solution: We will prove by induction on r. If $r = 1$, then we are done. Assume the result for $r-1$ and $I \subseteq \bigcup_{i=1}^{r} P_i$. If I is contained in an $r-1$ union of P_i's, then by applying induction one can conclude that I is contained in one of them. Suppose, if possible, for each i, $I \nsubseteq \bigcup_{j=1, j\neq i}^{r} P_j$. Choose $x_i \in I$, for all $1 \leq i \leq r$, such that $x_i \notin P_j$, for any $j \neq i$. Since $I \subseteq \cup_{i=1}^{r} P_i$ and $x_i \notin P_j$ for $j \neq i$, $x_i \in P_i$, for each i. Let $z = \sum_{i=1}^{r} x_1 \cdots x_{i-1} x_{i+1} \cdots x_r$. Then $z \in I \setminus \bigcup_{i=1}^{r} P_i$, this a contradiction. Therefore $I \subseteq P_i$ for some $1 \leq i \leq r$.

(2) Find the nilradical of $\mathbb{Z}_{36}$ and $\mathbb{Z}_9$.

Solution: Recall that the nilradical of ring A is the intersection of all the prime ideals of A. If P is a prime ideal of $\mathbb{Z}_n$, then $\mathbb{Z}_n/P$ is a finite integral domain, so it is a field, and hence P is a maximal ideal. We only need to find the maximal ideals of $\mathbb{Z}_n$. We know that P is a maximal ideal of $\mathbb{Z}_n$ if and only if $P = p\mathbb{Z}_n$ for some prime divisor p of n. Therefore the maximal ideals of $\mathbb{Z}_{36}$ are $2\mathbb{Z}_{36}$, $3\mathbb{Z}_{36}$ and the maximal ideal of $\mathbb{Z}_9$ is $3\mathbb{Z}_9$. Hence $\text{nil}(\mathbb{Z}_{36}) = 2\mathbb{Z}_{36} \cap 3\mathbb{Z}_{36} = 6\mathbb{Z}_{36}$ and $\text{nil}(\mathbb{Z}_9) = 3\mathbb{Z}_9$.

(3) Let F be a field and let $R = F[x, y]$. If $I = (x^2, xy)$ and $S = \{x\}$, then compute $I : S$.

Solution: Clearly x and y are in $I : S$. Let $f \in I : S$. By definition $fx \in I$. Let

$$
\begin{aligned}
fx &= \alpha_1 x^2 + \alpha_2 xy, \text{where } \alpha_1, \alpha_2 \in R. \\
x(f &- (\alpha_1 x + \alpha_2 y)) = 0.
\end{aligned}
$$

Since R is integral domain, $f = \alpha_1 x + \alpha_2 y$. Hence $f \in (x, y)$ and $I : S = (x, y)$.

(4) If M is an A-module, then prove that $\text{Hom}_A(A, M) \cong M$.

Solution: Define a map $\phi : \text{Hom}_A(A, M) \to M$ by $\phi(f) = f(1)$. Then for $f, g \in \text{Hom}_A(A, M)$, $\phi(f + g) = (f + g)(1) = f(1) + g(1) = \phi(f) + \phi(g)$. Also, for $\alpha \in A$ and $f \in \text{Hom}_A(A, M)$, $\phi(\alpha f) = \alpha f(1) = \alpha\phi(f)$. Therefore ϕ is an A-module homomorphism. For each $m \in M$, let $f_m : A \to M$ be the map $f_m(a) = am$. Then $\phi(f_m) = f_m(1) = m$. Therefore ϕ is surjective. Suppose $\phi(f) = f(1) = 0$. Then for any $a \in A$, $f(a) = f(a.1) = af(1) = 0$, i.e., $f = 0$. Therefore ϕ is an isomorphism.