NPTEL COURSE - Introduction to Commutative Algebra

Assignment 2

(1) Let S be a set and $\mathcal{F}(S) := \{f : S \to \mathbb{R} \mid f \text{ is a function }\}$ with pointwise addition and pointwise multiplication. Prove that $\mathcal{F}(S)$ is a ring.
Solution: The map $O(x) = 0$ is the additive identiy of $\mathcal{F}(S)$, since for each $f \in \mathcal{F}(S)$, $(O + f)(x) = O(x) + f(x) = 0 + f(x) = f(x)$ and $(f + O)(x) = f(x) + 0 = f(x)$.
For each $f \in \mathcal{F}(S)$, let $g(x) = -f(x)$. Then $(f + g)(x) = f(x) + g(x) = f(x) + -f(x) = 0 = O(x)$. Therefore g is the additive inverse of f.
For $f, g, h \in \mathcal{F}(S)$, $[(f + g) + h](x) = (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) = [f + (g + h)](x)$. Therefore, the addition is associative.
If $f, g \in \mathcal{F}(S)$, then for $x \in S$, $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$. Therefore, $f + g = g + f$. Therefore $\mathcal{F}(S)$ is an abelian group.
For $f, g, h \in \mathcal{F}(S)$, $[f(gh)](x) = f(x)(g(x)h(x)) = (f(x)g(x))h(x) = [(fg)h](x)$. Therefore, the multiplication is associative.
For $f, g, h \in \mathcal{F}(S)$, $[(f + g)h](x) = (f + g)(x)(h(x) = (f(x) + g(x))h(x) = f(x)h(x) + g(x)h(x) = [fh + gh](x)$. Therefore, the multiplication is distributive over the addition.
Hence $\mathcal{F}(S)$ is a ring. In fact, $\mathcal{F}(S)$ is a commutative ring, since for $f, g \in \mathcal{F}(S)$, $(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x)$.

(2) If A and B are rings and $f : A \to B$ is a ring homomorphism, then prove that $\ker f$ is an ideal of A.
Solution: If $a, b \in \ker f$, then $f(a + b) = f(a) + f(b) = 0 + 0 = 0$. Hence $a + b \in \ker f$. Since by the property of homomorphism $f(0) = 0$, $0 \in \ker f$. If $a \in \ker f$, then $f(-a) = -f(a) = 0$. Therefore, $-a \in \ker f$. Since A is an abelian group $a + b = b + a$ is always satisfied, in particular for $a, b \in I$. Therefore I is an abelian group. Now, if $a \in I$ and $b \in A$, then $f(ab) = f(a)f(b) = 0 \cdot f(b) = 0$. Hence $ab \in \ker f$. Therefore, $\ker f$ is an ideal of A.

(3) Let A be a ring. Prove that $I = \{p(x) \in A[x] \mid p(0) = 0\}$ is an ideal of $A[x]$.
Solution: Define a map $\phi : A[x] \to A$ by $\phi(p(x)) = p(0)$. Since $[p(x) + q(x)]_{x=0} = p(0) + q(0)$ and $[p(x)q(x)]_{x=0} = p(0)q(0)$, ϕ is a homomorphism.

Now,

$$\begin{aligned}\ker\phi &= \{p(x)\in A[x] \mid \phi(p(x))=0\}\\ &= \{p(x)\in A[x] \mid p(0)=0\}\\ &= I.\end{aligned}$$

Therefore, I is kernel of a homomorphism. Hence I is an ideal.

(4) Let A be a ring and $\mathfrak{p}$ be a prime ideal of A. Prove that $\mathfrak{P}=\{\sum_{i=0}^{n} a_i x^i \in A[x] \mid a_i \in \mathfrak{p}\}$ is a prime ideal of $A[x]$.
Solution: Let $B = A/\mathfrak{p}$ and let $\phi : A[x] \to B[x]$ be the homomorphism $\phi(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \bar{a}_i x^i$, where $\bar{a}_i$ denote the coset of a_i in $A/\mathfrak{p}$. A polynomial is zero if and only if all its coefficients are zero. Therefore

$$\begin{aligned}\ker\phi &= \left\{\sum_{i=0}^{n} a_i x^i \in A[x] \mid \bar{a}_i = 0\right\}\\ &= \left\{\sum_{i=0}^{n} a_i x^i \in A[x] \mid a_i \in \mathfrak{p}\right\}\\ &= \mathfrak{P}.\end{aligned}$$

We have shown that $\mathfrak{P}$ is kernel of a homomorphism. Hence $\mathfrak{P}$ is an ideal. Moreover, since ϕ is surjective, by the first isomorphism theorem, $A[x]/\mathfrak{P} \cong B[x]$. Since $\mathfrak{p}$ is a prime ideal, $B = A/\mathfrak{p}$ is an integral domain and hence $B[x]$ is an integral domain. Therefore $\mathfrak{P}$ is a prime ideal.

(5) Prove that $\mathbb{Z}_9$ is a local ring. Write down the maximal ideal of $\mathbb{Z}_9$.
Solution: An element $\bar{n}$ is a unit in $\mathbb{Z}_9$ if and only if $gcd(n, 9) = 1$. Therefore, the set of all non-units of $\mathbb{Z}_9$ is $\{\bar{0}, \bar{3}, \bar{6}\}$. This is an ideal in $\mathbb{Z}_9$. Hence $\mathbb{Z}_9$ is a local ring with the maximal ideal $\{\bar{0}, \bar{3}, \bar{6}\}$.