NPTEL COURSE - Introduction to Commutative Algebra

Assignment - Week 12

(1) Let A be a PID and I be a non-zero ideal in A. Prove that A/I is an Artinian ring.

(2) Let k be a field and X, Y, Z be variables. Set $R = k[X, Y, Z]/(X^2 - Y^3 - 1, XZ - 1)$ and let $x, y, z \in R$ be the images of X, Y, Z respectively. Set $t := x + z$. Let $P = k[t]$. Prove that x, y are integral over P.

(3) Let k be an algebraically closed field and J be an ideal of $k[x_1, \ldots, x_n]$. Let $f \in k[x_1, \ldots, x_n]$ be such that $f(P) = 0$ for all $P \in V(J)$ and $J' = J + (fY - 1) \subset k[x_1, \ldots, x_n, Y]$. Then
  (a) $V(J') = \emptyset$;
  (b) Using (a), prove that $f \in \text{rad}(J)$;
  (c) Conclude that $I(V(J)) = \text{rad}(J)$.