NPTEL COURSE - Introduction to Commutative Algebra

Assignment - Week 12

(1) Let A be a PID and I be a non-zero ideal in A. Prove that A/I is an Artinian ring.

Solution. *Recall.* A ring A is Artin $\Longleftrightarrow$ A is Noetherian and $\dim(A) = 0$.
Since A is PID, A is Noetherian ring. Note that $\dim(A/I) = 0$. Therefore A/I is an Artinian ring.

(2) Let k be a field and X, Y, Z be variables. Set $R = k[X, Y, Z]/(X^2 - Y^3 - 1, XZ - 1)$ and let $x, y, z \in R$ be the images of X, Y, Z respectively. Fix $a, b \in k$ and set $t := x + ay + bz$. Let $P = k[t]$. Prove that x, y are integral over P.

Solution: First of all, note that R is integral over $k[x, z]$ since $y^3 = x^2 - 1$. Also, since $xz = 1$, x as well as z satisfy the eqution $T^2 - (x + z)T + 1$ over $k[t]$. Therefore, $k[x, z]$ is integral over $k[x + z]$. Therefore R is integral over $k[x + z] = k[t]$.

(3) Let k be an algebraically closed field and J be an ideal of $k[x_1, \ldots, x_n]$. Let $f \in k[x_1, \ldots, x_n]$ be such that $f(P) = 0$ for all $P \in V(J)$ and $J' = J + (fY - 1) \subset k[x_1, \ldots, x_n, Y]$. Then

(a) $V(J') = \emptyset$;

Solution. Let

$$V(J') = \{(\alpha_1, \ldots, \alpha_n, \beta) \in k^{n+1} \mid g(\alpha_1, \ldots, \alpha_n, \beta) = 0, \text{ for all } g \in J'\}.$$

We need to show that $V(J') = \emptyset$. Suppose $(\alpha_1, \ldots, \alpha_n, \beta) \in V(J')$. Since $J \subseteq J'$, $(\alpha_1, \ldots, \alpha_n) \in V(J)$. By given hypothesis $f(\alpha_1, \ldots, \alpha_n) = 0$. Since $fY - 1 \in J'$,

$$\begin{aligned} f(\alpha_1, \ldots, \alpha_n)\beta - 1 &= 0 \\ 0.\beta - 1 &= 0. \end{aligned}$$

This is not possible. Therefore $V(J') = \emptyset$.

(b) Using (a), prove that $f \in \operatorname{rad}(J)$;

Solution. Since $V(J') = \emptyset$, $J' = (1) = k[x_1, \ldots, x_n, Y]$. Now $1 \in J'$, implies that there exist $p_i(x_1, \ldots, x_n, Y)$, $Q(x_1, \ldots, x_n, Y) \in k[x_1, \ldots, x_n, Y]$ and $h_i(x_1, \ldots, x_n) \in J$ such that

$$1 = \sum_{\text{finite sum}} p_i h_i + (-1 + yf)Q[x_1, \ldots, x_n, Y].$$

Since f is non-zero polynomial, $Y = f(x_1, \ldots, x_n)^{-1} \in k(x_1, \ldots, x_n)$

$$\begin{aligned}
1 &= \sum p_i(x_1, \ldots, x_n) f^{-1} h_i(x_1, \ldots, x_n). \\
&= \frac{\sum g_i(x_1, \ldots, x_n) h_i(x_1, \ldots, x_n)}{f^k} \\
f^k &= \sum g_i(x_1, \ldots, x_n) h_i(x_1, \ldots, x_n). \\
f^k &\in J, \text{ therefore} f \in \text{rad}(J).
\end{aligned}$$

(c) Conclude that $I(V(J)) = \text{rad}(J)$.

Solution. Clearly, $I(V(J)) \supseteq \text{rad}(J)$. From (b), we can conclude that, $I(V(J)) \subseteq \text{rad}(J)$. Therefore, $I(V(J)) = \text{rad}(J)$.