NPTEL COURSE - Introduction to Commutative Algebra

Assignment - Week 11

(1) Let A be a ring and M be an A-module. Prove that

$$\mathrm{Ass}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{Hom}_A(A/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0\}.$$

Solution. We have $\mathrm{Hom}(A/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0$ if and only if there is a $\phi \in \mathrm{Hom}(A/\mathfrak{p}, M)$ with $(0 : \phi) \subseteq \mathfrak{p}$. If $\mathfrak{p} \in \mathrm{Ass}(M)$, then there is an injective homomorphism $\phi : A/\mathfrak{p} \to M$. Therefore $\phi \in \mathrm{Hom}(A/\mathfrak{p}, M)$ and $(0 : \phi) \subseteq \mathfrak{p}$. Hence $\mathrm{Hom}(A/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0$.

Suppose there is a $\phi \in \mathrm{Hom}(A/\mathfrak{p}, M)$ with $(0 : \phi) \subseteq \mathfrak{p}$. This means, there is a non zero-element m in M such that $(0 : m) = \mathfrak{p}$. Therefore $\mathfrak{p} \in \mathrm{Ass}(M)$.

(2) For an ideal $I \subset A$, prove that if $I = \mathrm{rad}(I)$, then I has no embedded primary components.

Solution. Let $I = \mathrm{rad}(I) = \cap_{i\in\lambda}\mathfrak{p}_i$, where λ is any indexed set. With out loss of generality, we assume that decomposition is minimal, i.e., for all i, $\mathfrak{p}_i \not\supseteq \cap_{j\neq i}\mathfrak{p}_j$ and $\mathfrak{p}_i \neq \mathfrak{p}_j$ if $i \neq j$. By minimality of primary decomposition, I has no embedded primary components.

(3) Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an ideal in A. Prove that $\cap_{n=1}^{\infty} I^n = 0$.

Solution. Let $J = \cap_{n=1}^{\infty} I^n$. First we claim that $IJ = J$. Let $IJ = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cdots Q_k$, where Q_i is $\mathfrak{p}_i$-primary, $\mathfrak{p}_i \supset I$ for $1 \leq i \leq r$ and $\mathfrak{p}_j \not\supset I$ for $r < j \leq k$. If $y_j \in I \setminus \mathfrak{p}_j$ for $r < j \leq k$, then $x \in J \;\Rightarrow\; xy_j \in IJ \;\Rightarrow\; xy_j \in Q_j \;\Rightarrow\; x \in Q_j$. Therefore, $J \subseteq Q_{r+1} \cap \cdots \cap Q_k$. Since $I \subseteq \mathfrak{p}_i$ for $1 \leq i \leq r$, $I^m \subseteq Q_1 \cap \cdots \cap Q_r$ for some $m \gg 0$. Since $J \subseteq I^m$, we have $J \subseteq Q_1 \cap \cdots Q_r$. Therefore $J \subseteq Q_1 \cap \cdots \cap Q_k = IJ \;\Rightarrow\; J = IJ$. Since A is a local ring, I is contained in the Jacobson radical and hence by Nakayama's lemma, $J = 0$.

(4) Prove that a finitely generated $\mathbb{Z}$-module M is Artinian if and only if M is finite.

Solution. If M is finite $\mathbb{Z}$-module, then M is Artinian. Assume that M is finitely generated Artinian $\mathbb{Z}$-module. Therefore,

$$M \cong \mathbb{Z}^f \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z},$$

where $\mathbb{Z}^f$ is free $\mathbb{Z}$-module and $d_1|\cdots|d_k$. Since M is an Artinian, $\mathbb{Z}^f = 0$. Hence M is finite.