NPTEL COURSE - Introduction to Commutative Algebra

Assignment solution- Week 10

(1) Let A be a Noetherian ring, B a finitely generated A-algebra, G a finite group of A-automorphisms of B and $B^G := \{x \in B \mid f(x) = x \text{ for all } f \in G\}$. Show that B^G is a finitely generated A-algebra.

Solution. Since B^G is closed under addition and multiplication, it is a subring of B. We claim that B is integral over B^G. Let $b \in B$ and consider the polynomial in t:

$$f(t) = \prod_{\lambda \in G}(t - \lambda b).$$

Clearly $f(b) = 0$. The action of G on B extends in a natural way to an action of G on $B[t]$, where G acts trivially on t and acts on the coefficients as the action on B. For each $\sigma \in G$,

$$\sigma(f(t)) = \prod_{\lambda \in G}(t - \sigma(\lambda b)) = \prod_{\lambda \in G}(t - \lambda b) = f(t).$$

It follows that all the coefficients of $f(t)$ are G-invariant, and so lie in B^G. Since $f(t)$ is monic polynomial, b is integral over B^G. Hence the claim. Since B is a finitely generated A-algebra and A is Noetherian, any A-subalgebra is finitely generated. Hence B^G is a finitely generated A-algebra.

(2) If $n\mathbb{Z} \subset \mathbb{Z}$ is an irreducible ideal, then prove that $n = p^r$ for some prime p and a positive integer r.

Solution. Let $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_\ell^{\beta_\ell}$ be the prime decomposition of n. If $\ell > 1$, then $n\mathbb{Z} \subset p_i^{\beta_i}\mathbb{Z}$, for all $1 \leq i \leq \ell$. Moreover, since p_i's are co-prime, $n\mathbb{Z} = \cap_i^\ell p_i^{\beta_i}$. This contradicts the assumption that $n\mathbb{Z}$ is an irreducible ideal. Hence $\ell = 1$ and hence $n = p_1^{\beta_1}$.

(3) Find a minimal primary decomposition of $(x^3, x^2y^2, xz^3) \subset k[x, y, z]$. List the isolated and embedded prime ideals.

Solution. Using the result that if $ab \in I$ is a minimal generator such that $\gcd(a, b) = 1$, then $I = [I' + (a)] \cap [I' + (b)]$, where I' is the ideal generated by a minimal generating set of I without the element ab, we get

$$\begin{aligned}(x^3, x^2y^2, xz^3) &= (x^3, x^2y^2, x) \cap (x^3, x^2y^2, z^3) \\ &= (x) \cap (x^3, x^2, z^3) \cap (x^3, y^2, z^3) \\ &= (x) \cap (x^2, z^3) \cap (x^3, y^2, z^3).\end{aligned}$$

Therefore isolated prime ideal is (x) and embedded prime ideals are (x, z) and (x, y, z).