Assignment 5: solutions

1. If $f(x,y) = (-y,x)$ and $g(x,y) = (\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}})$, then the composition of g with itself produces f, i.e., $g \circ g = f$. $\boxed{True}$

2. Let f be an arbitrary affine transformation of $\mathbb{R}^2$. Let $f(1,1) = (p,q)$. Then $f(2,2) = (2p, 2q)$. $\boxed{False}$

   Take f to be a translation; say $f(x,y) = (x+1, y+1)$. This does not satisfy the given relation.

3. Let f be an arbitrary linear transformation of $\mathbb{R}^2$. Let $f(1,1) = (p,q)$. Then $f(2,2) = (2p, 2q)$. $\boxed{True}$

4. Let f be an arbitrary linear transformation of $\mathbb{R}^2$. The image of the unit circle $x^2 + y^2 = 1$ under f is a circle. $\boxed{False}$

   It could also be an ellipse.

5. There is a unique linear transformation of $\mathbb{R}^2$ which maps the X-axis to the line $y = 2x$, and the Y-axis to the line $y = x$. $\boxed{False}$

   Any linear transformation of the form $f(x,y) = (ax + by, 2ax + by)$ has this property, where a, b are arbitrary real numbers. Thus, there are infinitely many such linear transformations.

6. Let S be the square with vertices $(0,0), (1,0), (0,1), (1,1)$. The number of linear transformations of $\mathbb{R}^2$ which map S to itself is:

   - 1
   - 2
   - 3
   - infinitely many.

   Let f be a linear transformation which maps S to itself. Then, each vertex of S maps to a vertex of S. It is easy to see that $f(0,0) = (0,0)$, and $f(1,1)$ can only be $(1,1)$. Thus, there is only freedom in where $(1,0)$ and $(0,1)$ map to. We can either have $f(1,0) = (1,0)$, $f(0,1) = (0,1)$ or $f(1,0) = (0,1)$, $f(0,1) = (1,0)$. These correspond to the linear transformations $f(x,y) = (x,y)$ and $f(x,y) = (y,x)$.

7. Let $a > 0$ and define the linear transformation $f(x,y) = (ax - y, ax + y)$. If f dilates areas of regions of $\mathbb{R}^2$ by a factor of 6, then the value of a is: $\boxed{3}$

   The matrix representation of f is $\begin{pmatrix} a & -1 \\ a & 1 \end{pmatrix}$. Its determinant is $-2a$. The area dilation factor of f is the absolute value of this determinant, which is $2a$ (since $a > 0$). Thus $2a = 6$, or $a = 3$.