Assignment 4: Solutions

1. If $f(x)$ is a polynomial of degree 2 such that (i) $f(0) > 0$, (ii) $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and (iii) $f(0) = f(1)$, $f(\frac{1}{2}) = 0$. Then, the smallest possible value of $f(2)$ is: $\boxed{9}$

   Condition (ii) implies $f(x)$ can be written as

   $$f(x) = a\frac{x(x-1)}{2} + bx + c$$

   where $a, b, c \in \mathbb{Z}$. Condition (i) implies $c \geq 1$, and condition (iii) implies that $b = 0$ and $a = 8c$. Thus, $f(x) = c\,(4x(x-1)+1)$, and so $f(2) = 9c$. The least value occurs when $c = 1$.

2. If $f(x)$ is a polynomial of degree ≤ 2 such that $f(x) \in \mathbb{Z}$ for $x = 0, 1, 2$. Then $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

   - True.
   - False.

   Any polynomial $f(x)$ of degree ≤ 2 can be written as

   $$f(x) = a\frac{x(x-1)}{2} + bx + c$$

   where $a, b, c \in \mathbb{R}$. Now, $f(x) \in \mathbb{Z}$ for $x = 0, 1, 2$ implies that c, $b + c$ and $a + 2b + c$ are all integers. This clearly implies that a, b, c are integers, and hence that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

3. If $f(x)$ is a polynomial of degree ≤ 2 such that $f(x) \in \mathbb{Z}$ for $x = 0, 2, 4$. Then $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

   - True.
   - False.

   Take for instance $f(x) = x/2$.

4. If $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, then we will call $f(x)$ an integral polynomial. Suppose f, g are integral polynomials, then which of the folllowing are integral polynomials ?

   - $f(x) + g(x)$.
   - $f(x)g(x)$.
   - $f(g(x))$.
   - $\frac{f(x)(g(x)-1)}{2}$.

   This follows from the definition.

5. If $f(x) + g(x)$ is an integral polynomial, then so are $f(x)$ and $g(x)$.

   - True.
   - False.

   Take $f(x) = g(x) = x/2$.

6. If $f(x)g(x)$ is an integral polynomial, then so are $f(x)$ and $g(x)$.

   - True.
   - False.

   Take $f(x) = x$, $g(x) = \frac{x-1}{2}$.

7. If $f(g(x))$ is an integral polynomial, then so are $f(x)$ and $g(x)$.

   - True.
   - False.

   Take $f(x) = 2x$ and $g(x) = x/2$.