Assignment 1: Solutions

In multiple choice questions, the correct answer is marked in red.

1. Let f, g be polynomials with $\deg f = 3, \deg g \leq 3$. Suppose $f(x) = g(x)$ for $x = 1, 2, 3, 4$. What is the degree of g ?

   - 0
   - 1
   - 2
   - 3
   - cannot say.

   Since g agrees with f at 4 points, and both f, g have degree at most 3, we must have $f = g$; thus the degree of g is also 3.

2. Let f be a polynomial such that $f(1) = 2, f(2) = 7, f(3) = 9$. The remainder obtained on division of $f(x)$ by $x - 2$ is: $\boxed{7}$.

   More generally, the remainder on division of $f(x)$ by $x - a$ is $f(a)$ for any real (or complex) number a.

3. Let f be a polynomial such that $f(n) = 2^n$ for $n = 0, 1, 2, 3$. What is the minimum possible value of the degree of f ? $\boxed{3}$.

   Let $g(x)$ be the polynomial obtained using Lagrange interpolation with the conditions $g(n) = 2^n$ for $n = 0, 1, 2, 3$. First compute g and check that g has degree 3. Next, let f be any polynomial such that $f(n) = 2^n$ for $n = 0, 1, 2, 3$. Let $h(x) = f(x) - g(x)$; then $h(n) = 0$ for $n = 0, 1, 2, 3$. Hence

   $$h(x) = x(x-1)(x-2)(x-3)\, q(x) \textit{ for some polynomial } q(x).$$

   So either $h(x) = 0$ or $\deg h \geq 4$. Thus $f(x) = g(x) + h(x)$ has degree ≥ 3, and with equality exactly when $f(x) = g(x)$.

4. Let f be a polynomial such that $f(n) = \frac{n(n-1)}{2}$ for $n = 0, 1, 2, 3$. What is the minimum possible value of the degree of f ? $\boxed{2}$

   Same reasoning as in the previous problem, except now, the interpolating polynomial $g(x)$ turns out to have degree 2.

5. Let f, g, h be polynomials of degree ≤ 1. Let $k(x) = f(x)g(x^2)h(x^4)$. If $\deg k = 5$, what is the value of $\deg f + 3\deg g + 9\deg h$? $\boxed{10}$

   The given conditions imply that $\deg f = 1, \deg g = 0, \deg h = 1$.

6. Let f be a polynomial of degree ≤ 2 satisfying $f(1) = 1, f'(1) = 2, f''(1) = 3$. What is the value of $f(1)$? $\boxed{3}$

   Use Taylor's formula (the version where the derivatives are given at a point other than 0; see "problem to think about" in the Taylor's formula video).

7. Let $A = (1, 1)$ and $B = (1, 1)$ be points in $\mathbb{R}^2$. Let D be the point in $\mathbb{R}^2$ such that the vector $\vec{OD}$ satisfies $\vec{OD} \cdot \vec{OA} = 3$ and $\vec{OD} \cdot \vec{OB} = -1$. What is the value of $\vec{OD} \cdot \vec{OD}$? $\boxed{5}$

   One computes that $D = (1, 2)$.

8. Let f, g be polynomials of degrees a, b respectively. Let $h(x) = f(g(x))$. The degree of h is:

   - ab
   - $a + b$
   - a^b
   - b^a

9. Let f, g, h be nonzero polynomials such that $f(x) - g(x) = h(x)$ and $\deg f = \deg h$. Pick the true statement:

   - $\deg g \leq \deg f$.
   - $\deg g > \deg f$.
   - $\deg g$ has no relation to $\deg f$.

10. Let f, g, h be polynomials such that $f(x) = g(x) + x^3\, h(x)$. Then $f^{(j)}(0) = g^{(j)}(0)$ for

    - $j = 0$.
    - $j = 1$.
    - $j = 2$.
    - all the above.