Assignment - 1

Curves AND surfaces **July 19, 2016**

1. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a regular curve. Show that arc length $s(t) = \int_a^t \|\dot{\gamma}(u)\| du,\ t \in [a,b]$ is independent of parametrization. Show that $s(t)$ is a smooth function of t.

2. Show that reparametrization of a regular curve is regular.

3. Show that the curve $\gamma : (-1,1) \to \mathbb{R}^2$ given by $g(t) = (t^3, t^6)$ does not have a unit speed parametrization. Find unit speed paramatrization for the circle $(x - x_0)^2 + (y - y_0)^2 = R^2$.

4.
   (a) Show that any regular plane curve whose curvature is a positive constant is a part of a circle.
   (b) Suppose a unit speed curve has signed curvature $k_S(s) = s$. Find the equation of the curve. (The resulting curve is called Cornu's spiral).

5.
   (a) Let γ be a unit speed plane curve with no where vanishing curvature. Define the centre of curvature $\varepsilon(s)$ of γ at a point $\gamma(s)$ to be $\varepsilon(s) = \gamma(s) + \frac{1}{k_S(s)}\eta_S(s)$. Prove that the circle with centre at $\varepsilon(s)$ and radius $\frac{1}{k_S(s)}$ is tangent to γ at $\gamma(s)$ and has the same curvature there.
   (b) With above notation, consider $\varepsilon(s)$ as parametrisation of a new curve called the evolute of γ. Calculate the signed curvature of ε.

6. A string of length l is attached to the point $s = 0$ of a unit speed plane curve γ. The string is wound on to the curve while being kept taught. Find the curve traced out by its other endpoint. (The resulting curve is called involute of γ).

7. For a space curve with no where vanishing curvature show that $\tau(s) = \frac{(\dot{\gamma}(s)\times\ddot{\gamma}(s))\bullet\dddot{\gamma}(s)}{\|\dot{\gamma}(s)\times\ddot{\gamma}(s)\|^2}$. ($\times$ denote the cross product and $\bullet$ denotes the dot product in $\mathbb{R}^3$).

8. Describe all curves in $\mathbb{R}^3$ which have constant curvature $k > 0$ and constant torsion τ.