Assignment - 3

Differential Calculus in Several Variables **March 29**

1. Express $x^2 + xy + y^2 =$ in powers of $(x-1)$ and $(y-2)$. (Use Taylor's formula).

2. Let f be real-valued and assume that the directional derivatives $D_u f(x+tu)$ exits for each $t \in [0,1]$. Prove that for some $\theta \in (0,1)$ we have $f(x+u) - f(x) = D_u f(x+\theta u)$.

3. If f is real-valued and if the directional derivatives $D_u f(x) = 0$ for every x in an open ball $B(x,\delta)$ and every directions u, prove that f is constant on $B(x,\delta)$.

4. Investigate the following functions for maxima and minima or saddle points.

(a) $f : \mathbb{R}^3 \to \mathbb{R},\ f(x,y,z) = x^2y + y^2z + z^2 - 2x$.

(b) $f : \mathbb{R}^3 \to \mathbb{R},\ f(x,y,z) = (ax^2 + by^2 + cz^2)e^{-x^2-y^2-z^2}$.

5. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, U open, has continuous first and second order partial derivatives and $X \in U$ is a stationary point for f. Let $H = ((\frac{\partial^2 f_i}{\partial x_j \partial x_i}))$ be the second derivative of f at X. Denote by H_k the kth principal minor of H.

Prove the following:

(a) If $\det H_k < 0$ for some $k = 1, 2, \ldots, n$ then X is a saddle point.

(b) If $\det H \neq 0$ then

(i) f has a local minimum at X if and only if $\det H_k > 0$ for all k.

(ii) f has a local minimum at X if and only if $(-1)^k \det H_k > 0$ for all k.

(iii) f has a saddle point at X if and only if it is neither local maximum or minimum.

(c) If $\det H = 0$ then the test is inconclusive (give an example).

6. Find the shortest distance from the point $(a,0)$ to the parabola $y^2 + 4x = 0$.

7. Find the point on the line of intersection of the two planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ which is nearest to the origin.