Assignment - 2

Differential Calculus in Several Variables **March 21, 2016**

1. Let $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^m$. Write $f = (f_1, f_2, \cdots, f_m)$ where $f_i : U \to \mathbb{R}$. Prove that f is differentiable at $X_0 \in \text{int}\, U$ if and only if each f_i is differentiable at X_0. Express $Jf(X_0)$ in terms of $\nabla f_i(X_0),\ i = 1, 2, \cdots, m$.

2. Let f, g be differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Assume that f is differentiable at $X_0,\ f(X_0) = 0$ and g is continuous at X_0. Let $h(x) = g(X) \cdot f(X)$. Prove that h is differentiable at X_0 and compute its derivative.

3. Let $f(x, y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y)$. Compute the Jacobian.

4. Prove that given a point $X_0 \in \mathbb{R}^n$ there is no real valued function on $\mathbb{R}^n$ such that $D_u f(X_0) > 0$ for every direction $u \in \mathbb{R}^n$. Give an example such that $D_u f(X_0) > 0$ for a fixed direction u and any point X_0.

5. Let f be real-valued function differentiable at $X_0 \in \mathbb{R}^n$ and $\|\nabla f(X_0)\| \neq 0$. Show that there is one and only one vector Y_0 such that $|Df_{X_0}(Y_0)| = \|\nabla f(X_0)\| \neq 0$.

6. Compute $\nabla f(x, y)$ at those points in $\mathbb{R}^2$ where it exists.

(a) $f(x,y) = \begin{cases} x^2y^2 \log(x^2+y^2) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}.$

(b) $f(x,y) = \begin{cases} xy \sin \frac{1}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}.$

7. Given $f : \mathbb{R}^2 \to \mathbb{R}$ define $F(r, \theta) = f(r\cos\theta, r\sin\theta)$. Compute first and second order partial derivatives of F in terms of those of f. Verify that $\|\nabla f(r\cos\theta, r\sin\theta)\|^2 = [\frac{\partial F}{\partial x}(r,\theta)]^2 + \frac{1}{r^2}\frac{\partial F}{\partial y}(r,\theta)]^2$.

8. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable at each point of $\mathbb{R}^2$. Let g_1, g_2 are defined from $\mathbb{R}^3$ to $\mathbb{R}$ by

$$g_1(x,y,z) = x^2 + y^2 + z^2, \quad g_2(x,y,z) = x + y + z.$$

Let $g : \mathbb{R}^3 \to \mathbb{R}^2$, $g(x,y,z) = (g_(x,y,z), g_2(x,y,z))$ and $h = f \circ g$. Show that

$$\|\nabla h\|^2 = 4(\frac{\partial f}{\partial x_1})^2 g_1 + 4(\frac{\partial f}{\partial x_1})(\frac{\partial f}{\partial x_2}) g_2 + 3(\frac{\partial f}{\partial x_2})^2.$$

9. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ satisfies $f(\lambda X) = \lambda f(X)$ for every $X \in U$, every $\lambda \in \mathbb{R}$ and some $p > 0$ such that $\lambda X \in U$. If f is differentiable at $X_0 \in U$ show that $X \dot{\nabla} f(X_0) = pf(X)$.

10. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(t) = (\cos t, \sin t)$. Show that the mean value formula $f(y) - f(x) = f'(z)(y - x)$ does not hold.

11. Show that any open connected set in $\mathbb{R}^n$ is polygonally connected; meaning, if $U \subseteq \mathbb{R}^n$ is open connected and $X, Y \in U$ then there exists $Z_0 = X, Z_1, Z_2, \cdots Z_k = Y$ such that X and Y can be connected by lines starting from Z_0 to Z_1 then Z_1 to Z_2 so on and finally Z_{n-1} to Z_n.