Assignment - 1

Differential Calculus in Several Variables **March 14, 2016**

A column vector $X \in \mathbb{R}^n$ is denoted by $X = (x_1, x_2, \cdots, x_n)'$.

1. Here we define another 'notion' of distance between two points $X = (x_1, x_2, \cdots, x_n)', Y = (y_1, y_2, \cdots, y_n)' \in \mathbb{R}^n$ by

$$d_1(X,Y) = \sum_{i=1}^{n} |x_i - y_i|.$$

Let $d(X,Y)$ be the usual Euclidean distance we considered in Lecture-1. Show that d_1 and d are equivalent in the sense that there exist constants A, B (calculate A and B explicitly) such that for any two points $X, Y \in \mathbb{R}^n$ we have

$$Ad_1(X,Y) \leq d(X,Y) \leq Bd_1(X,Y).$$

2. Let $A \subseteq \mathbb{R}^n$ be a compact set and $f : A \to \mathbb{R}^m$ be continuous. Show that $f(A)$ is compact.

3. Show that connected subsets of $\mathbb{R}$ are intervals. Show that any convex set in $\mathbb{R}^n$ is connected.

4. Show that the following functions are continuous from $\mathbb{R}^2$ to $\mathbb{R}$.

(a) $f(x,y) = \begin{cases} \frac{\sin(xy)}{\sqrt{x^2+y^2}} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

(b)

(c) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}.$

5. Show that the function $f(x,y) = \begin{cases} \frac{x^4y^4}{(x^2+y^4)^3} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$ approaches the origin along any straight line but NOT continuous at origin.

6. Compute the matrix of the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^3$ given by $A(x,y)' = (x, x-y, x+y)'$ with respect to the basis $(1,0)', (0,1)'$ of $\mathbb{R}^2$ and basis $(1,0,0)', (1,0,1)', (1,1,1)'$ of $\mathbb{R}^3$.

7. Calculate the first order partial derivatives and directional derivatives for the following real-valued functions defined on $\mathbb{R}^3$.

(a) $f(x) = \|x\|^4$.

(b) $f(x) = x \cdot L(x)$ where $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear function.

(c) $f(x) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j$ where $a_{ij} = a_{ji}$.

8. Let $f_1, f_2, \cdots, f_n$ are real valued differentiable functions from (a, b). For each $x = (x_1, x_2, \cdots, x_n)$ where $a < x_i < b, i = 1, 2, \cdots n$ define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = f_1(x_1) + f_2(x_2) \cdots + f_n(x_n)$. Show that f is differentiable at each point of its domain and also calculate the directional derivatives.