Question 1:

a)

In this case $S_0 = 0.80$, $K = 0.81$, $r = 0.08$, $r_f = 0.05$, $\sigma = 0.15$, $T = 0.5833$

$$d_1 = \frac{\ln(0.80/0.81) + (0.08 - 0.05 + 0.15^2/2) \times 0.5833}{0.15\sqrt{0.5833}} = 0.1016$$
$$d_2 = d_1 - 0.15\sqrt{0.5833} = -0.0130$$

$$N(d_1) = 0.5405; \quad N(d_2) = 0.4998$$

The delta of one call option is $e^{-r_f T}N(d_1) = e^{-0.05\times0.5833} \times 0.5405 = 0.5250$.

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2} = \frac{1}{\sqrt{2\pi}}e^{-0.00516} = 0.3969$$

so that the gamma of one call option is

$$\frac{N'(d_1)e^{-r_f T}}{S_0\sigma\sqrt{T}} = \frac{0.3969 \times 0.9713}{0.80 \times 0.15 \times \sqrt{0.5833}} = 4.206$$

The vega of one call option is

$$S_0\sqrt{T}N'(d_1)e^{-r_f T} = 0.80\sqrt{0.5833} \times 0.3969 \times 0.9713 = 0.2355$$

The theta of one call option is

$$\begin{aligned} & -\frac{S_0 N'(d_1)\sigma e^{-r_f T}}{2\sqrt{T}} + r_f S_0 N(d_1)e^{-r_f T} - rKe^{-rT}N(d_2) \\ = & -\frac{0.8 \times 0.3969 \times 0.15 \times 0.9713}{2\sqrt{0.5833}} \\ & + 0.05 \times 0.8 \times 0.5405 \times 0.9713 - 0.08 \times 0.81 \times 0.9544 \times 0.4948 \\ = & -0.0399 \end{aligned}$$

The rho of one call option is

$$\begin{aligned} & KTe^{-rT}N(d_2) \\ = & 0.81 \times 0.5833 \times 0.9544 \times 0.4948 \\ = & 0.2231 \end{aligned}$$

Delta can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the value of an option to buy one yen increases by 0.525 times that amount. Gamma can be interpreted as meaning that, when the spot price increases

b)

The fund is worth \$300,000 times the value of the index. When the value of the portfolio falls by 5% (to \$342 million), the value of the S&P 500 also falls by 5% to 1140. The fund manager therefore requires European put options on 300,000 times the S&P 500 with exercise price 1140.

(a) $S_0 = 1200$, $K = 1140$, $r = 0.06$, $\sigma = 0.30$, $T = 0.50$ and $q = 0.03$. Hence:

$$d_1 = \frac{\ln(1200/1140) + \left(0.06 - 0.03 + 0.3^2/2\right) \times 0.5}{0.3\sqrt{0.5}} = 0.4186$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.2064$$

$$N(d_1) = 0.6622; \quad N(d_2) = 0.5818$$

$$N(-d_1) = 0.3378; \quad N(-d_2) = 0.4182$$

The value of one put option is

$$\begin{aligned} &1140e^{-rT}N(-d_2) - 1200e^{-qT}N(-d_1) \\ =&1140e^{-0.06\times0.5} \times 0.4182 - 1200e^{-0.03\times0.5} \times 0.3378 \\ =&63.40 \end{aligned}$$

The total cost of the insurance is therefore

$$300,000 \times 63.40 = \$19,020,000$$

(b) From put–call parity

$$S_0e^{-qT} + p = c + Ke^{-rT}$$

or:

$$p = c - S_0e^{-qT} + Ke^{-rT}$$

This shows that a put option can be created by selling (or shorting) e^{-qT} of the index, buying a call option and investing the remainder at the risk-free rate of interest. Applying this to the situation under consideration, the fund manager should:

1) Sell $360e^{-0.03\times0.5} = \$354.64$ million of stock
2) Buy call options on 300,000 times the S&P 500 with exercise price 1140 and maturity in six months.
3) Invest the remaining cash at the risk-free interest rate of 6% per annum.

This strategy gives the same result as buying put options directly.

(c) The delta of one put option is

$$\begin{aligned} &e^{-qT}[N(d_1) - 1] \\ =&e^{-0.03\times0.5}(0.6622 - 1) \\ =&- 0.3327 \end{aligned}$$

This indicates that 33.27% of the portfolio (i.e., \$119.77 million) should be initially sold and invested in risk-free securities.

(d) The delta of a nine-month index futures contract is

$$e^{(r-q)T} = e^{0.03\times0.75} = 1.023$$

The spot short position required is

$$\frac{119,770,000}{1200} = 99,808$$

times the index. Hence a short position in

$$\frac{99,808}{1.023 \times 250} = 390$$

futures contracts is required.

Question #2:

a)

The delta of the portfolio is

$$-1,000 \times 0.50 - 500 \times 0.80 - 2,000 \times (-0.40) - 500 \times 0.70 = -450$$

The gamma of the portfolio is

$$-1,000 \times 2.2 - 500 \times 0.6 - 2,000 \times 1.3 - 500 \times 1.8 = -6,000$$

The vega of the portfolio is

$$-1,000 \times 1.8 - 500 \times 0.2 - 2,000 \times 0.7 - 500 \times 1.4 = -4,000$$

(a) A long position in 4,000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4,000 \times 1.5 = +6,000$. The delta of the whole portfolio (including traded options) is then:

$$4,000 \times 0.6 - 450 = 1,950$$

Hence, in addition to the 4,000 traded options, a short position in £1,950 is necessary so that the portfolio is both gamma and delta neutral.

(b) A long position in 5,000 traded options will give a vega-neutral portfolio since the long position has a vega of $5,000 \times 0.8 = +4,000$. The delta of the whole portfolio (including traded options) is then

$$5,000 \times 0.6 - 450 = 2,550$$

Hence, in addition to the 5,000 traded options, a short position in £2,550 is necessary so that the portfolio is both vega and delta neutral.

b)

In this case $S_0 = 0.85$, $K = 0.85$, $r = 0.05$, $r_f = 0.04$, $\sigma = 0.04$ and $T = 0.75$. The option can be valued using equation (15.11)

$$d_1 = \frac{\ln(0.85/0.85) + (0.05 - 0.04 + 0.0016/2) \times 0.75}{0.04\sqrt{0.75}} = 0.2338$$
$$d_2 = d_1 - 0.04\sqrt{0.75} = 0.1992$$

and

$$N(d_1) = 0.5924, \quad N(d_2) = 0.5789$$

The value of the call, c, is given by

$$c = 0.85e^{-0.04\times 0.75} \times 0.0.5924 - 0.85e^{-0.05\times 0.75} \times 0.5789 = 0.0147$$

i.e., it is 1.47 cents. From put–call parity

$$p + S_0e^{-r_fT} = c + Ke^{-rT}$$

so that

$$p = 0.0147 + 0.85e^{-0.05\times 9/12} - 0.85e^{-0.04\times 9/12} = 0.00854$$

Question #3:

a)

Answer: (C)

This problem is based on Exercise 14.21 on page 465 of McDonald (2006).

Let S_1 denote the stock price at the end of one year. Apply the Black-Scholes formula to calculate the price of the at-the-money call one year from today, conditioning on S_1.

$d_1 = [\ln(S_1/S_1) + (r + \sigma^2/2)T]/(\sigma\sqrt{T}) = (r + \sigma^2/2)/\sigma = 0.41667$, which turns out to be independent of S_1.

$d_2 = d_1 - \sigma\sqrt{T} = d_1 - \sigma = 0.11667$

The value of the forward start option at time 1 is

$C(S_1) = S_1N(d_1) - S_1e^{-r}N(d_2)$
$= S_1[N(0.41667) - e^{-0.08}N(0.11667)]$
$= S_1[0.66154 - e^{-0.08}\times 0.54644]$
$= 0.157112S_1$.

(Note that, when viewed from time 0, S_1 is a random variable.)

Thus, the time-0 price of the forward start option must be 0.157112 multiplied by the time-0 price of a security that gives S_1 as payoff at time 1, i.e., multiplied by the prepaid forward price $F_{0,1}^P(S)$. Hence, the time-0 price of the forward start option is

$$0.157112\times F_{0,1}^P(S) = 0.157112\times e^{-0.08}\times F_{0,1}(S) = 0.157112\times e^{-0.08}\times 100 = 14.5033$$

b)

The prices are not arbitrage-free. To show that Mary's portfolio yields arbitrage profit,

	Time 0	Time T			
		$S_T < 40$	$40 \le S_T < 50$	$50 \le S_T < 55$	$S_T \ge 55$
Buy 1 call Strike 40	-11	0	$S_T - 40$	$S_T - 40$	$S_T - 40$
Sell 3 calls Strike 50	$+18$	0	0	$-3(S_T - 50)$	$-3(S_T - 50)$
Lend \$1	-1	e^{rT}	e^{rT}	e^{rT}	e^{rT}
Buy 2 calls strike 55	-6	0	0	0	$2(S_T - 55)$
Total	0	$e^{rT} > 0$	$e^{rT} + S_T - 40 > 0$	$e^{rT} + 2(55 - S_T) > 0$	$e^{rT} > 0$

Peter's portfolio makes arbitrage profit, because:

	Time-0 cash flow	Time-T cash flow
Buy 2 calls & sells 2 puts Strike 55	$2(-3 + 11) = 16$	$2(S_T - 55)$
Buy 1 call & sell 1 put Strike 40	$-11 + 3 = -8$	$S_T - 40$
Lend \$2	-2	$2e^{rT}$
Sell 3 calls & buy 3 puts Strike 50	$3(6 - 8) = -6$	$3(50 - S_T)$
Total	0	$2e^{rT}$

4)
a) (i)
b) (ii)
c) (i)
d) (i)
e) (iv)