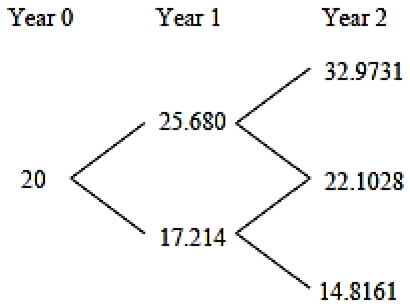
1)a)

First, we construct the two-period binomial tree for the stock price.



The calculations for the stock prices at various nodes are as follows:

$S_u = 20 \times 1.2840 = 25.680$
$S_d = 20 \times 0.8607 = 17.214$
$S_{uu} = 25.68 \times 1.2840 = 32.9731$
$S_{ud} = S_{du} = 17.214 \times 1.2840 = 22.1028$
$S_{dd} = 17.214 \times 0.8607 = 14.8161$

The risk-neutral probability for the stock price to go up is

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502.$$

Thus, the risk-neutral probability for the stock price to go down is 0.5498.

If the option is exercised at time 2, the value of the call would be
$C_{uu} = (32.9731 - 22)_+ = 10.9731$
$C_{ud} = (22.1028 - 22)_+ = 0.1028$
$C_{dd} = (14.8161 - 22)_+ = 0$

If the option is European, then $C_u = e^{-0.05}[0.4502C_{uu} + 0.5498C_{ud}] = 4.7530$ and $C_d = e^{-0.05}[0.4502C_{ud} + 0.5498C_{dd}] = 0.0440$.
But since the option is American, we should compare C_u and C_d with the value of the option if it is exercised at time 1, which is 3.68 and 0, respectively. Since $3.68 < 4.7530$ and $0 < 0.0440$, it is not optimal to exercise the option at time 1 whether the stock is in the up or down state. Thus the value of the option at time 1 is either 4.7530 or 0.0440.

Finally, the value of the call is
$C = e^{-0.05}[0.4502(4.7530) + 0.5498(0.0440)] = 2.0585.$

1)b)

(a) The probability distributions are:

$$\phi(2.0+0.1,\ 0.16) = \phi(2.1, 0.16)$$

$$\phi(2.0+0.6,\ 0.16\times 6) = \phi(2.6,\ 0.96)$$

$$\phi(2.0+1.2,\ 0.16\times 12) = \phi(3.2,\ 1.96)$$

(b) The chance of a random sample from $\phi(2.6,\ 0.96)$ being negative is

$$N\left(-\frac{2.6}{\sqrt{0.96}}\right) = N(-2.65)$$

where $N(x)$ is the cumulative probability that a standardized normal variable [i.e., a variable with probability distribution $\phi(0,\ 1)$] is less than x. From normal distribution tables $N(-2.65) = 0.0040$. Hence the probability of a negative cash position at the end of six months is 0.40%.
Similarly the probability of a negative cash position at the end of one year is

$$N\left(-\frac{3.2}{\sqrt{1.96}}\right) = N(-2.30) = 0.0107$$

or 1.07%.

(c) In general the probability distribution of the cash position at the end of x months is

$$\phi(2.0+0.1x,\ 0.16x)$$

The probability of the cash position being negative is maximized when:

$$\frac{2.0+0.1x}{\sqrt{0.16x}}$$

is minimized. Define

$$y = \frac{2.0+0.1x}{0.4\sqrt{x}} = 5x^{-\frac{1}{2}} + 0.25x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -2.5x^{-\frac{3}{2}} + 0.125x^{-\frac{1}{2}}$$

$$= x^{-\frac{3}{2}}(-2.5+0.125x)$$

2)a)

The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

In this case

$$B = \frac{1}{x}$$

so that:

$$\frac{\partial B}{\partial t} = 0; \quad \frac{\partial B}{\partial x} = -\frac{1}{x^2}; \quad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}$$

Hence

$$\begin{aligned} dB &= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{1}{2}s^2x^2\frac{2}{x^3}\right]dt - \frac{1}{x^2}sxdz \\ &= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right]dt - \frac{s}{x}dz \end{aligned}$$

The expected instantaneous rate at which capital gains are earned from the bond is therefore:

$$-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

The expected interest per unit time is 1. The total expected instantaneous return is therefore:

$$1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

When expressed as a proportion of the bond price this is:

$$\left(1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right) / \left(\frac{1}{x}\right)$$

$$= x - \frac{a}{x}(x_0 - x) + s^2$$

2)b)

A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in two years? (Hint $S_T > 80$ when $\ln S_T > \ln 80$.)

The variable $\ln S_T$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and standard deviation $\sigma\sqrt{T}$. In this case $S_0 = 50$, $\mu = 0.12$, $T = 2$, and $\sigma = 0.30$ so that the mean and standard deviation of $\ln S_T$ are $\ln 50 + (0.12 - 0.3^2/2)2 = 4.062$ and $0.3\sqrt{2} = 0.424$, respectively. Also, $\ln 80 = 4.382$. The probability that $S_T > 80$ is the same as the probability that $\ln S_T > 4.382$. This is

$$1 - N\left(\frac{4.382 - 4.062}{0.424}\right) = 1 - N(0.754)$$

where $N(x)$ is the probability that a normally distributed variable with mean zero and standard deviation 1 is less than x. From the tables at the back of the book $N(0.754) = 0.775$ so that the required probability is 0.225.

3)a)

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278$$

and

$$Ke^{-rT} = 40e^{-0.05} = 38.049$$

Hence, if the option is a European call, its value c is given by

$$c = 42N(0.7693) - 38.049N(0.6278)$$

If the option is a European put, its value p is given by

$$p = 38.049N(-0.6278) - 42N(-0.7693)$$

Using the polynomial approximation just given or the NORMSDIST function in Excel,

$$N(0.7693) = 0.7791, \qquad N(-0.7693) = 0.2209$$

$$N(0.6278) = 0.7349, \qquad N(-0.6278) = 0.2651$$

so that

$$c = 4.76, \qquad p = 0.81$$

Ignoring the time value of money, the stock price has to rise by $2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by $2.81 for the purchaser of the put to break even.

3)b)

In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.397 or 39.7% per annum.

4)a)

$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$

$$K\left[1 - e^{-r(T-t_2)}\right] = 55(1 - e^{-0.08\times 0.4167}) = 1.80$$

Hence

$$D_2 < K\left[1 - e^{-r(T-t_2)}\right]$$

Also:

$$K\left[1 - e^{-r(t_2-t_1)}\right] = 55(1 - e^{-0.08\times 0.5}) = 2.16$$

Hence:

$$D_1 < K\left[1 - e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 13.12 that the option should never be exercised early.

The present value of the dividends is

$$1.5e^{-0.3333\times 0.08} + 1.5e^{-0.8333\times 0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times 1.25} \times 0.3692 = 4.17$$

or $4.17.

4)b)

In this case $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (13.3), given by:

$$\ln S_T \sim \phi[\ln 50 + (0.18 - \frac{0.09}{2})2,\ 0.3^2 \times 2]$$

i.e.,

$$\ln S_T \sim \phi(4.18,\ 0.18)$$

The mean stock price is from equation (13.4)

$$50e^{2\times 0.18} = 50e^{0.36} = 71.67$$

and the standard deviation is, from equation (13.5),

$$50e^{2\times 0.18}\sqrt{e^{0.09\times 2} - 1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96 \times 0.42 \quad \text{and} \quad 4.18 + 1.96 \times 0.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

5) The bias is estimated by av - 2; the square root of var estimates the standard error; a t statistic for testing the null is the estimated bias divided by the square root of its estimated variance, var/4000;

and the confidence interval is given by the interval between the 200th and the 3800th r values, adjusted for any bias